AMERICAN MATHEMATICAL SOCIETY

COLLOQUIUM PUBLICATIONS

VOLUME V, PART II

SECOND EDITION

1931

ANALYSIS SITUS

BY

OSWALD VEBLEN

HENRY B. FINE PROFESSOR OF MATHEMATICS
PRINCETON UNIVERSITY

SECOND EDITION

NEW YORK
PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
501 WEST 116TH STREET
1931

Photographically
reproduced 1951

EDWARDS BROTHERS, INC.
Lithoprinters
ANN ARBOR, MICHIGAN

PREFACE TO THE SECOND EDITION

The second edition is essentially a reprint of the first, prepared in response to the demand which still seems to exist for copies of the book and in the hope that it will be a useful adjunct to the more modern and extensive books on the same subject by my colleagues Lefschetz and Alexander. In correcting such errors as have come to attention the original text has been changed as little as possible, and the changes have not altered the point of view of the original. The chief changes are in the definition of oriented circuit, which has caused a rearrangement at the beginning of Chapter IV, and in the correction of the proof of the invariance of the coefficients of torsion in Chapter IV together with the corresponding discussion of the invariants of a group in Chapter V.

My thanks are due to several colleagues who have sent in corrections, particularly to Messrs. Alexander, Lefschetz, Morse, Nielsen and Pfeiffer. I owe most of all to Dr. A. B. Brown who has done most of the work of reconstructing faulty proofs and definitions and incorporating changes in the text in such a way as to preserve a self-consistent whole.

At the suggestion of Professor Lefschetz I am including as appendices at the end of the volume a paper on Intersection Numbers which was written as a part of the original book and the paper by Philip Franklin and myself on Matrices whose Elements are Integers which was intended for reference in this book.

AUTHOR'S PREFACE

The Cambridge Colloquium Lectures on Analysis Situs were intended as an introduction to the problem of discovering the n-dimensional manifolds and characterizing them by means of invariants. For the present publication the material of the lectures has been thoroughly revised and is presented in a more formal way. It thus constitutes something like a systematic treatise on the elements of Analysis Situs. The author does not, however, imagine that it is in any sense a definitive treatment. For the subject is still in such a state that the best welcome which can be offered to any comprehensive treatment is to wish it a speedy obsolescence.

The definition of a manifold which has been used is that which proceeds from the consideration of a generalized polyhedron consisting of n-dimensional cells. The relations among the cells are described by means of matrices of integers and the properties of the manifolds are obtained by operations with the matrices. The most important of these matrices were introduced by H. Poincaré to whom we owe most of our knowledge of n-dimensional manifolds* for the cases in which $n > 2$. But it is also found convenient to employ certain more elementary matrices of incidence whose elements are reduced modulo 2, and from which the Poincaré matrices can be derived.

The operations on the matrices lead to combinatorial results which are independent of the particular way in which a manifold is divided into cells and therefore lead to theorems of

* Poincaré's work is contained in the following four memoirs: Analysis Situs, Journal de l'École Polytechnique, 2d Ser., Vol. 1 (1895); Complément à l'Analysis Situs, Rendiconti del Circolo Matematico di Palermo, Vol. 13 (1899); Second Complément, Proceedings of the London Mathematical Society, Vol. 32 (1900); Cinquième Complément, Rendiconti, Vol. 18 (1904). The third and fourth Complements deal with applications to Algebraic Geometry, into which we do not go.

Analysis Situs. The proof that this is so is based on an article by J. W. Alexander in the Transactions of the American Mathematical Society, Vol. 16 (1915), p. 148. The continuous transformations and the singularities (in the way of overlapping, etc.) which are allowed in this proof are completely general, so that we are able to avoid the difficulties, foreign to Analysis Situs, which beset those treatments of the subject which restrict attention to analytic transformations or singularities.

It will be seen that, aside from this one question which has to be dealt with in order to give significance to the combinatorial treatment, we leave out of consideration all the work that has been done on the point-set problems of Analysis Situs and on its foundation in terms of axioms or definitions other than those actually used in the text. We have also been obliged by lack of space to leave out all reference to the applications. We have not even given a definition of an n-cell by means of a set of equations and inequalities, or the discussion of orientation by means of the signs of determinants. These are to be found in very readable form in Poincaré's first paper, where they are given as the basis of his work. They belong properly, however, to the applications of the subject. For in nearly all cases when Poincaré (or anyone else) has proved a theorem of Analysis Situs, he has been obliged to set up a machinery which is equivalent to a set of matrices.

No attempt has been made to give a complete account of the history and literature of the subject. These are covered for the period up to 1907 by the article on Analysis Situs by Dehn and Heegard in the Encyklopädie (Vol. III_1, p. 153); and the more important works subsequent to that date which bear on our part of the subject are referred to in Chap. V.

I take pleasure in acknowledging my indebtedness to Professor J. W. Alexander who has read the manuscript and made many valuable suggestions, and also to Dr. Philip Franklin who has helped with the manuscript, the drawings, and the proof-sheets.

PRINCETON, MAY, 1921.

CONTENTS

CHAPTER I

LINEAR GRAPHS

Sections		Pages
1.	Fundamental Definitions	1
5.	Order Relations on Curves	3
7.	Singular Complexes	5
10.	The Simplest Invariants	7
14.	Symbols for Sets of Cells	9
16.	The Matrices H_0 and H_1	11
18.	Zero-dimensional Circuits	13
22.	One-dimensional Circuits	16
26.	Trees	18
28.	Geometric Interpretation of Matrix Products	19
30.	Reduction of H_0 and H_1 to Normal Form	21
33.	Oriented Cells	23
36.	Matrices of Orientation	25
41.	Oriented 1-Circuits	28
43.	Symbols for Oriented Complexes	29
47.	Normal Form for E_0	32
48.	Matrices of Integers	32
49.	Normal Form for E_1	33

CHAPTER II

TWO-DIMENSIONAL COMPLEXES AND MANIFOLDS

1.	Fundamental Definitions	36
4.	Matrices of Incidence	37
7.	Subdivision of 2-Cells	39
11.	Maps	42
13.	Regular Subdivision	43
17.	Manifolds and 2-Circuits	46

Sections | Pages

28. The Connectivity R_1 52
32. Singular Complexes 55
35. Bounding and Non-bounding 1-Circuits 57
37. Congruences and Homologies, Modulo 2 58
39. The Correspondence A 60
47. Invariance of R_1 64
50. Invariance of the 2-Circuit 66
54. Matrices of Orientation 68
57. Orientable Circuits 70
61. Normal Forms for Manifolds 74

CHAPTER III

COMPLEXES AND MANIFOLDS OF n DIMENSIONS

1. Fundamental Definitions 76
4. Matrices of Incidence 78
9. The Connectivities R_i 81
11. Reduction of the Matrices H_i to Normal Form 82
15. Congruences and Homologies, Modulo 2 84
17. Theory of the n-Cell 86
20. Regular Complexes 88
24. Manifolds 91
25. Dual Complexes 92
29. Duality of the Connectivities R_i 94
31. Generalized Manifolds 95
35. Bounding and Non-bounding Sets of k-Circuits 97
42. Invariance of the Connectivities R_i 102

CHAPTER IV

ORIENTABLE MANIFOLDS

1. Oriented n-Cells 104
3. Matrices of Orientation 105
4. Covering Oriented Complexes 105
6. Boundary of an Oriented Complex 107
8. Orientable n-Circuits 109
12. Oriented k-Circuits 112

Sections Pages
14. Normal Form of E_k 112
17. The Betti Numbers 117
20. The Coefficients of Torsion 119
22. Relation between the Betti Numbers and the Connectivities 120
25. Congruences and Homologies 120
27. The Fundamental Congruences and Homologies . . . 122
31. Bounding k-Circuits 125
35. Invariance of the Betti Numbers and Coefficients of Torsion 128
38. Duality of the Betti Numbers and Coefficients of Torsion 130

CHAPTER V

THE FUNDAMENTAL GROUP AND CERTAIN UNSOLVED PROBLEMS

1. Homotopic and Isotopic Deformations 132
3. Isotopy and Order Relations 133
10. The Indicatrix 136
13. Theorems on Homotopy 138
15. The Fundamental Group 139
18. The Group of a Linear Graph 141
22. The Group of a Two-dimensional Complex 143
25. The Commutative Group $\bar{G}$ 145
28. Equivalences and Homologies 148
30. The Poincaré Numbers of G 149
36. Covering Manifolds 152
38. Three-dimensional Manifolds 154
40. The Heegaard Diagram 155
44. The Knot Problem 158

APPENDIX I

The Intersection Numbers 159

APPENDIX II

On Matrices whose Elements are Integers 170

CHAPTER I

LINEAR GRAPHS

Fundamental Definitions

1. We shall presuppose a knowledge of some of the elementary properties of the real Euclidean space of n dimensions ($n \leqq 3$ for the first two chapters). In such a space, the points collinear with and between two distinct points constitute a *segment* or *one-dimensional simplex* whose *ends* or *vertices* are the given points. The ends are not regarded as points of the segment. For obvious reasons of symmetry, a single point will be referred to as a *0-dimensional simplex*.

2. Consider any set of objects in (1–1) correspondence* with the points of a segment and its two ends. The objects corresponding to the points of the segment constitute a *one-dimensional cell* or *1-cell* and those corresponding to the ends constitute the *ends* or *boundary* of the 1-cell. In like manner a single object may be referred to as a *0-cell*.

In the cases which are usually considered the objects which constitute a cell and its boundary are points of a k-space and the correspondence which defines the cell is continuous. Consequently a 1-cell is an arc of curve joining two distinct points. In the general case, however, it would be meaningless to say that the correspondence was continuous, because continuity implies previously determined order relations, and here the order relations of a cell are determined by means of the defining correspondence.

The objects which constitute a cell and its boundary will always be referred to as "points" in the following pages.

---

* By (1–1) correspondence we mean a correspondence which is one-to-one reciprocal; i. e., a (1–1) correspondence between two sets [A] and [B] is such that each A corresponds to one and only one B and each B is the correspondent of one and only one A.

The order relations among any set of points on the cell or its boundary are by definition identical with those of the corresponding points of the segment and its boundary. Hence, in particular, a point P is a *limit point* of a set of points $[X]$ of a cell and its boundary if and only if the corresponding point P of the segment is a limit point of the corresponding set of points $[X]$ of the segment and its boundary.

A *continuous transformation* of a cell and its boundary into itself or into another cell and its boundary is now defined

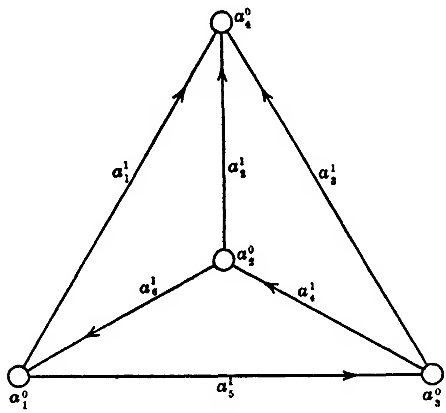

FIG. 1.

as a transformation of the cell and its boundary which if it carries a set $[X]$ to a set $[X']$ carries every limit point of $[X]$ to a limit point of $[X']$.

3. A *zero-dimensional complex* is a set of distinct 0-cells, finite in number. A *one-dimensional complex* or a *linear graph* is a zero-dimensional complex together with a finite number of 1-cells bounded by pairs of its 0-cells, such that no two of the 1-cells have a point in common and each 0-cell is an end of at least one 1-cell. Let us denote the number of 0-cells by α_0 and the number of 1-cells by α_1. The 0-cells are sometimes called *vertices* and the 1-cells *edges*.

For example, the vertices and edges of a tetrahedron (Fig. 1) constitute a linear graph for which $\alpha_0 = 4$ and $\alpha_1 = 6$. A linear graph is not necessarily assumed to lie in any space, being defined in a purely abstract way. It is obvious, however, that if α_0 points be chosen arbitrarily in a Euclidean three-space they can be joined by pairs in any manner whatever by α_1 non-intersecting simple arcs. Therefore, any linear graph may be thought of as situated in a Euclidean three-space.

For some purposes it is desirable to use the term one-dimensional complex to denote a more general set of 1-cells and 0-cells than that described above. For example, a 1-cell and its two ends form a one-dimensional complex according to the definition above, but a 1-cell by itself or a 1-cell and one of its ends do not. In the following pages we shall occasionally refer to an arbitrary subset of the 1-cells and 0-cells of a linear graph as a *generalized one-dimensional complex.*

4. A transformation F of a set of points $[X]$ of a complex C_1 into a set of points $[X']$ of the same or another complex is said to be *continuous* if and only if it is continuous in the sense of § 2 on each complex composed of a 1-cell of C_1 and its ends (i. e., if the transformation effected by F on those X's which are on such a 1-cell and its ends is continuous). A (1–1) continuous transformation of a complex into itself or another complex is called, following Poincaré, a *homeomorphism.* The inverse transformation is easily proved to be continuous. Two complexes related by a homeomorphism are said to be *homeomorphic.*

The set of all homeomorphisms by which a linear graph is carried into itself obviously forms a group. Any theorem about a linear graph which states a property which is left invariant by all transformations of this group is a theorem of *one-dimensional Analysis Situs.* The group of homeomorphisms of a linear graph is its *Analysis Situs group.*

Order Relations on Curves

5. By an *open curve* is meant the set of all points of a complex composed of a 1-cell and its two ends. By

a *closed curve* is meant the set of all points of a complex C_1 consisting of two distinct 0-cells a_1^0, a_2^0 and two 1-cells a_1^1, a_2^1, each of which has a_1^0 and a_2^0 as ends but which have no common points (Fig. 2). The most elementary theorems about curves are those which codify the order relations. They may be stated (without proof) as follows:

Let us denote a 1-cell and its ends by a^1, a_1^0 and a_2^0. If a_3^0 is any point af a^1, there are two 1-cells a_1^1 and a_2^1 such that a_1^1 has a_1^0 and a_3^0 as its ends, a_2^1 has a_3^0 and a_2^0 as its ends, and every point of a^1 is either on a_1^1 or a_2^1 or identical with a_3^0. The 1-cell a^1 is said to be *separated* into the 1-cells a_1^1 and a_2^1 by the 0-cell a_3^0.

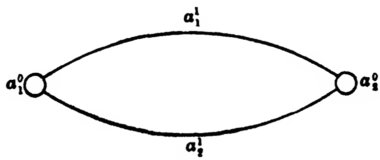

Fig. 2.

A 0-cell is said to be *incident* with a 1-cell if and only if it is an end of the 1-cell; and under the same conditions the 1-cell is said to be incident with the 0-cell. It follows directly from the theorem on separation in the paragraph above that n distinct points of the 1-cell a^1 determine $n+1$ 1-cells such that the n points (or 0-cells) may be denoted by $b_1^0, b_2^0, \cdots, b_n^0$ and the $n+1$ 1-cells by $b_1^1, b_2^1, \cdots, b_{n+1}^1$ in such a way that each cell is incident with the cell which directly precedes or directly follows it in the sequence $a_1^0, b_1^1, b_1^0, b_2^1, \cdots, b_n^0, b_{n+1}^1, a_2^0$.

If $b_1^0, b_2^0, \cdots, b_n^0$ are n distinct points of a closed curve, the remaining points of the curve constitute n 1-cells b_i^1 $(i = 1, 2, \cdots, n)$, no two of which have a point in common, such that each b_i^0 is incident with just two of them.

6. A little reflection will convince the reader that many of the theorems about functions of one real variable and

about linear sets of points belong to one-dimensional Analysis Situs. As an example we may cite the theorem that any nowhere dense perfect set of points on a closed curve can be transformed into any other such set by a (1–1) continuous transformation of the curve. The Heine-Borel theorem is another case in point.

The theorems of Analysis Situs may be divided somewhat roughly into two classes, those dealing essentially with continuity considerations (of which the theorem on perfect sets of points cited above may serve as an illustration), and those having an essentially combinatorial character. It is the theorems of the latter class which will occupy most of our attention in the following pages, though we shall continually make use of theorems of the former class without proving them.

Singular Complexes

7. Let F be a correspondence between a 0-dimensional complex C_0 and a set of points $[P]$ of any complex C (for the present, C is 0- or 1-dimensional) in which each point of C_0 corresponds to a single P and each P is the correspondent of one or more points of C_0. The object obtained by associating any point X of C_0 with the point P which is its image under F will be denoted by $F(X)$ and called a point *on* C; it is said to *coincide* with P and P to coincide with it. The set of all points $F(X)$ on C is called a 0-dimensional complex *on* C. If any P is the correspondent of more than one point X of C_0, P is called a *singular point* and the complex on C is said to be *singular*.

8. Let C_1 be a generalized one-dimensional complex and let F be a continuous correspondence between C_1 and a set of points $[P]$ of a complex C, in which each point of C_1 corresponds to a single P and each P is the correspondent of at least one point of C_1. The object obtained by associating any point X of C_1 with the point P which is its image under this correspondence will be called a *point on* C and is uniquely denoted by the functional notation $F(X)$; it is said to *coincide* with P and P is said to *coincide* with it. The point $F(X_1)$

is called a limit point of the points $F(X)$ if X_1 is a limit point of the points X. The set of all points $F(X)$ on C is in a (1–1) continuous correspondence with the points of C_1 and thus constitutes a one-dimensional complex C_1' identical in structure with C_1. The one-dimensional complex C_1' is said to be *on* C. If any of the points P is the correspondent under F of more than one point of C_1, C_1' is called a *singular complex on* C and the point P in question a *singular point*. If the correspondence F is (1–1), C_1' is said to be *non-singular*.

It is to be emphasized that in the definitions above F is a perfectly general continuous function. Thus, for example, all the points of a 1-cell of C_1 may be imaged on a single point of C. In the rest of this chapter we shall be referring to non-singular complexes more often than to singular ones. We shall therefore understand that a complex is non-singular unless the opposite is stated.

9. Let P be any point of a generalized one-dimensional complex C_1. If P is a point of a 1-cell of C_1 let Q_1 and Q_2 be two points of this 1-cell such that P is between them. If P is a vertex, let $Q_1, Q_2, \cdots. Q_j$ be a set of points, one on each 1-cell of which P is an end. The set of points composed of P and of all points between P and the points Q_1, $Q_2, \cdots, Q_j$ is called a *neighborhood* of P.

A generalized one-dimensional complex C_1' which is on C_1 is said to *cover* C_1 in case there is at least one point of C_1' on each point of C_1 and there exists for every point of C_1' a neighborhood which is a non-singular complex on C_1. In case the number of points of C_1' which coincide with a given point of C_1 is finite and equal to n for every point of C_1, C_1' is said to *cover* C_1 *n times*.

The only connected complex which can cover a 1-cell is a 1-cell, or a subdivision of a 1-cell such as is described in § 5, and it can cover it only once. *A closed curve,* on the other hand, *can be covered any number of times by another closed curve.*

The truth of the latter statement may be seen very simply as follows. Let C_1 and C_1' be two circles in a Euclidean plane. Denote any point on C_1 by a coordinate θ $(0 < \theta \leq 2\pi)$, and

any point on C_1' by $\theta'\,(0<\theta'\leqq 2\pi)$. Let each point, θ, of C_1 correspond to the n points

$$\theta'=\frac{\theta}{n},\quad \theta'=\frac{2\theta}{n},\quad \cdots,\quad \theta'=\frac{(n-1)\theta}{n},\quad \theta'=\theta,$$

of C_1'. In case $n=2$, for example, a pair of opposite points of C_1' corresponds to a single point of C_1.

The Simplest Invariants

10. One of the first objects of Analysis Situs is to find the numerical invariants of complexes under the group of homeomorphisms. By an invariant under this group we mean a number $I(C)$ determined by a complex C in such a way that if C' be any complex homeomorphic with C, the number $I(C')$ determined in the same way for C' is the same as $I(C)$.

11. Starting with any point O of a complex C_1 consider all points of C_1 which can be joined to this one by open curves, singular or not,* on C_1. This set of points will contain all points of a certain set of 0-cells and 1-cells of C_1 (a sub-complex of C_1) which we may call C_1'. Since any two points of C_1' can be joined to O by open curves, they can be joined to each other by an open curve. Hence the same set of points is determined if any other point of C_1' replace O in the definition of C_1'.

Since C_1 is composed of a finite number of 0-cells and 1-cells altogether, it is composed of a finite number of sub-complexes defined in the same way that C_1' is defined in the paragraph above. The number of these sub-complexes contained in C_1 is obviously an invariant in the sense defined in § 10, for if two complexes C_1 and C_1' are homeomorphic, any curve on C_1 corresponds to a curve on C_1'. *This number shall be denoted by R_0.* If $R_0=1$, C_1 is said to be *connected.*

* No generality is gained by allowing the curves to be singular, but the argument is slightly easier, and more in the spirit of its generalizations to n dimensions.

12. Let us denote the number of 0-cells in a complex C_1 by α_0 and the number of 1-cells by α_1. *The number* $\alpha_0 - \alpha_1$ *is an invariant.*

To prove this, let us first observe that if C_1 be modified by introducing any point of one of its 1-cells as a 0-cell and thereby separating the 1-cell into two 1-cells, the number $\alpha_0 - \alpha_1$ is unchanged. For α_0 is changed to $\alpha_0 + 1$ and α_1 is changed to $\alpha_1 + 1$.

Now consider two linear graphs C_1 and C_1' between which there is a (1–1) continuous correspondence F. Suppose that C_1 has α_0 0-cells and α_1 1-cells and C_1' has α_0' 0-cells and α_1' 1-cells. Each 0-cell of C_1 which is an end of only one 1-cell will correspond under F to a 0-cell of C_1' having the same property; otherwise F could not be continuous. In like manner, each 0-cell of C_1 which is an end of more than two 1-cells will correspond to a 0-cell of C_1' which is an end of an equal number of 1-cells. For the same reasons, a 0-cell of C_1' which is an end of only one, or of more than two 1-cells is the correspondent of a like 0-cell of C_1.

A certain number of 0-cells of C_1 which are ends of two 1-cells each may correspond to points of C_1' which are not vertices. Suppose there are k such 0-cells of C_1 and therefore k corresponding points of C_1'. As explained above, any one of these points of C_1' may be introduced as a vertex, thereby changing C_1' into a complex with one more 0-cell and one more 1-cell. Repeating this step k times C_1' is changed into a complex C_1'' having $\alpha_0' + k$ 0-cells and $\alpha_1' + k$ 1-cells. The correspondence F will carry every vertex of C_1 into a vertex of C_1''.

Certain of the vertices of C_1'', however, may not be the correspondents under F of vertices of C_1. Suppose there are n such vertices of C_1''. By precisely the reasoning used in the last paragraph the points of C_1 which correspond to these n vertices of C_1'' may be introduced as vertices of C_1, converting C_1 into a complex $\overline{C}_1$ having $\alpha_0 + n$ 0-cells and $\alpha_1 + n$ 1-cells.

The complexes C_1'' and $\overline{C}_1$ have been defined so that under the (1–1) correspondence F each vertex of $\overline{C}_1$ corresponds

to a vertex of C_1'' and each 1-cell of $\bar{C}_1$ to a 1-cell of C_1''. Hence

$$\alpha_0 + n = \alpha_0' + k \quad \text{and} \quad \alpha_1 + n = \alpha_1' + k,$$

from which it follows that

$$\alpha_0 - \alpha_1 = \alpha_0' - \alpha_1'.$$

13. The invariant number $\alpha_0 - \alpha_1$ is called the *characteristic** of the linear graph. The number $\alpha_1 - \alpha_0 + R_0$ is called the *cyclomatic number*† and denoted by μ. In the case of a connected complex

$$\mu = \alpha_1 - \alpha_0 + 1.$$

The two invariants, R_0 and $\alpha_0 - \alpha_1$ are evidently not sufficient to characterize a linear graph completely. There is a rather elaborate theory of linear graphs‡ in existence which we shall not attempt to cover. Instead we shall go into detail on questions which cluster around the two invariants already found, because this part of the theory is the basis of important generalizations to n dimensions.

Symbols for Sets of Cells

14. Let us denote the 0-cells of a one-dimensional complex C_1 by $\overset{0}{a}_1, \overset{0}{a}_2, \cdots, \overset{0}{a}_{\alpha_0}$ and the 1-cells by $\overset{1}{a}_1, \overset{1}{a}_2, \cdots, \overset{1}{a}_{\alpha_1}$.

Any set of 0-cells of C_1 may be denoted by a symbol $(x_1, x_2, \cdots, x_{\alpha_0})$ in which $x_i = 1$ if $\overset{0}{a}_i$ is in the set and $x_i = 0$ if $\overset{0}{a}_i$ is not in the set. Thus, for example, the pair of points $\overset{0}{a}_1, \overset{0}{a}_4$ in Fig. 1 is denoted by $(1, 0, 0, 1)$. The total number of symbols $(x_1, x_2, \cdots, x_{\alpha_0})$ is 2^{α_0}. Hence the total number of sets of 0-cells, barring the 0-set, is $2^{\alpha_0} - 1$. The symbol for a null-set, $(0, 0, \cdots, 0)$ will be referred to as *zero* and denoted by 0.

* Cf. W. Dyck, *Math. Ann.*, Vol. 32, p. 457.

† The term is due to J. B. Listing, Census raumliche Komplexe, Gottingen, 1862. But the significance of this constant had been clearly brought out by G. Kirchhoff in the paper referred to in § 36 below.

‡ Cf. Dehn-Heegaard, Encyklopädie, III, AB, 3, pp. 172–178.

The marks 0 and 1 which appear in the symbols just defined, may profitably be regarded as residues, modulo 2, i. e., as symbols which may be combined algebraically according to the rules

$$0+0=1+1=0, 0+1=1+0=1, 0\times0=0\times1=1\times0=0, 1\times1=1.$$

Under this convention the *sum* (mod 2) of two symbols, or of the two sets of points which correspond to the symbols $(x_1, x_2, \cdots, x_{\alpha_0}) = X$ and $(y_1, y_2, \cdots, y_{\alpha_0}) = Y$, may be defined as $(x_1+y_1, x_2+y_2, \cdots, x_{\alpha_0}+y_{\alpha_0}) = X+Y$. Geometrically, $X+Y$ is the set of all points which are in X or in Y but not in both.*

For example, if $X=(1, 0, 0, 1)$ and $Y=(0, 1, 0, 1)$ $X+Y=(1, 1, 0, 0)$; i. e., X represents a_1^0 and a_4^0, Y represents a_2^0 and a_4^0, and $X+Y$ represents a_1^0 and a_2^0. Since a_4^0 appears in both X and Y, it is suppressed in forming the sum, modulo 2.

This type of addition has the obvious property that if two sets contain each an even number of 0-cells, the sum (mod. 2) contains an even number of 0-cells.

15. Any set, S, of 1-cells in C_1 may be denoted by a symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ in which $x_i = 1$ if a_i^1 is in the set and $x_i = 0$ if a_i^1 is not in the set. The 1-cells in the set may be thought of as labelled with 1's and those not in the set as labelled with 0's. The symbol is also regarded as representing the one-dimensional complex composed of the 1-cells of S and the 0-cells which bound them. Thus, for example, in Fig. 1 the boundaries of two of the faces are $(1, 0, 1, 0, 1, 0)$ and $(1, 1, 0, 0, 0, 1)$.

The sum (mod. 2) of two symbols $(x_1, x_2, \cdots, x_{\alpha_1})$ is defined in the same way as for the case of symbols representing 0-cells. Correspondingly if C_1' and C_1'' are one-dimensional complexes each of which is a sub-complex of a given one-dimensional complex C_1, the *sum*

* In other words, $X+Y$ is the difference between the logical sum and the logical product of the two sets of points. In terms of the logical operations, if S and S' are the given sets, this one is $S+S'-SS'$.

$$C_1' + C_1'' \text{ (mod. 2)}$$

is defined as the one-dimensional complex obtained by suppressing all 1-cells common to C_1' and C_1'' and retaining all 1-cells which appear only in C_1' or in C_1''. For example, in Fig. 1, the sum of the two curves represented by (1, 0, 1, 0, 1, 0) and (1, 1, 0, 0, 0, 1) is (0, 1, 1, 0, 1, 1) which represents the curve composed of a_2^1, a_3^1, a_5^1, a_6^1 and their ends.

The Matrices H_0 and H_1

16. It has been seen in § 11 that any one-dimensional complex falls into R_0 sub-complexes each of which is connected. Let us denote these sub-complexes by $C_1^1, C_1^2, \cdots, C_1^{R_0}$, and let the notation be assigned in such a way that a_i^0 $(i=1, 2, \cdots, m_1)$ are the 0-cells of C_1^1, a_i^0 $(i=m_1+1, \cdots, m_2)$ those of C_1^2, and so on.

With this choice of notation, the sets of vertices of C_1^1, $C_1^2, \cdots, C_1^{R_0}$, respectively, are represented by the symbols $(x_1, x_2, \cdots, x_{\alpha_0})$ which constitute the rows of the following matrix.

$$H_0 = \left\| \begin{array}{cccccccccccc} \multicolumn{4}{c}{\overbrace{\hphantom{1 \; 1 \; \cdots \; 1}}^{m_1}} & \multicolumn{4}{c}{\overbrace{\hphantom{0 \; 0 \; \cdots \; 0}}^{m_2 - m_1}} & & \multicolumn{4}{c}{\overbrace{\hphantom{0 \; 0 \; \cdots \; 0}}^{\alpha_0 - m_{R_0 - 1}}} \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & & & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & & & \cdot \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 0 \end{array} \right\| = \| \eta_{ij}^0 \| .$$

For most purposes it is sufficient to limit attention to connected complexes. In such cases $R_0 = 1$, and H_0 consists of one row all of whose elements are 1.

17. By the definition in § 5 a 0-cell is *incident* with a 1-cell if it is one of the ends of the 1-cell, and under the same conditions the 1-cell is incident with the 0-cell. The incidence relations between the 0-cells and 1-cells may be represented in a table or matrix of α_0 rows and α_1 columns as follows: The 0-cells of C_1 having been denoted by

a_i^0, $(i = 1, 2, \cdots, \alpha_0)$ and the 1-cells by a_j^1, $(j = 1, 2, \cdots, \alpha_1)$, let the element of the ith row and the jth column of the matrix be 1 if a_i^0 is incident with a_j^1 and let it be 0 if a_i^0 is not incident with a_j^1.

For example, the table for the linear graph of Fig. 1 formed by the vertices and edges of a tetrahedron is as follows:

	a_1^1	a_2^1	a_3^1	a_4^1	a_5^1	a_6^1
a_1^0	1	0	0	0	1	1
a_2^0	0	1	0	1	0	1
a_3^0	0	0	1	1	1	0
a_4^0	1	1	1	0	0	0

In the case of the complex used in § 5 to define a simple closed curve the incidence matrix is

$$\begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}.$$

We shall denote the element of the ith row and jth column of the matrix of incidence relations between the 0-cells and 1-cells by η_{ij}^1 and the matrix itself by

$$\| \eta_{ij}^1 \| = H_1.$$

The ith row of H_1 is the symbol for the set of all 1-cells incident with a_i^0 and the jth column is the symbol for the set of two 0-cells incident with a_j^1.

The condition which we have imposed on the graph, that both ends of every 1-cell shall be among the α_0 0-cells, implies that every column of the matrix contains exactly two 1's. Conversely, any matrix whose elements are 0's and 1's and which is such that each column contains exactly two 1's and each row contains at least one 1, can be regarded as the incidence matrix of a linear graph. For to obtain such a graph it is only necessary to take α_0 points in a 3-space, denote them arbitrarily by $a_1^0, a_2^0, \cdots, a_{\alpha_0}^0$, and join the pairs which correspond to 1's in the same column successively by arcs not meeting the arcs previously constructed.

This construction also makes it evident that there is a (1–1) continuous correspondence between any two graphs corresponding to the same matrix H_1.

Zero-dimensional Circuits

18. A pair of 0-cells is called a *0-dimensional circuit* or a *0-circuit* or a *0-dimensional manifold.* Any even number of 0-cells is a set of 0-circuits and the sum (mod. 2) of any number of 0-circuits is a set of 0-circuits.

If two 0-cells are the ends of an open curve on C_1 (cf. § 5) they are said to *bound* the open curve and *to be connected* by it. Such a pair of 0-cells is called a *bounding 0-circuit.* For example, in Fig. 1, a_1^0 and a_3^0 bound the curve a_5^1 and also bound the curve $a_1^1\, a_4^0\, a_3^1$.

19. In the symbol $(x_1, x_2, \cdots, x_{\alpha_0})$ for a bounding 0-circuit all the x's are 0 except two which correspond to a pair of vertices belonging to one of the connected complexes into which C_1 falls according to § 11. This symbol must therefore satisfy the following equations.

$$
(H_0)\qquad
\begin{array}{llcl}
x_1 & +x_2+\cdots+x_{m_1} & = & 0. \\
x_{m_1+1} & +\cdots+x_{m_2} & = & 0. \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & \\
x_{m_{R_0-1}+1} & +\cdots+x_{\alpha_0} & = & 0.
\end{array}
$$

in which the variables are reduced modulo 2, as explained in § 14. The matrix of these equations is H_0.

Since the symbol for any set of bounding 0-circuits is the sum (mod. 2) of the symbols for the 0-circuits of the set, it follows that any such symbol satisfies the equations (H_0). This is also evident because in the symbol for any set of bounding 0-circuits an even number of the x's in each of these equations must be 1. Hence any such symbol satisfies (H_0). On the other hand, the symbol for a non-bounding 0-circuit will not satisfy the equations (H_0) because the two x's which

are not zero in this symbol appear in different equations; and, in general, any set of vertices which is not a set of bounding 0-circuits will contain an odd number of vertices in some connected sub-complex of C_1, and hence its symbol will fail to satisfy these equations. *Hence the set of all solutions of* (H_0) *is the set of all symbols for sets of bounding* 0-*circuits.*

Since no two of these equations have a variable in common, they are linearly independent. Hence *all solutions of* (H_0) *are linearly dependent* (*mod.* 2) *on a set of* $\alpha_0 - R_0$ *linearly independent solutions.*

20. Denoting the connected sub-complexes of C_1 by C_1^1, $C_1^2, \cdots, C_1^{R_0}$ as in § 16 let the notation be so assigned that $a_1^1, \cdots, a_{m_1}^1$ are the 1-cells in C_1^1; $a_{m_1+1}^1, \cdots, a_{m_2}^1$ the 1-cells in C_1^2; and so on. The matrix H_1 then must take the form

$$\left\|\begin{matrix} \mathrm{I} & 0 & 0 & 0 & \\ 0 & \mathrm{II} & 0 & 0 & \\ 0 & 0 & \mathrm{III} & & \\ & & & & \end{matrix}\right\|$$

where all the non-zero elements are to be found in the matrices I, II, III, etc., and I is the matrix of C_1^1, II of C_1^2, etc. This is evident because no element of one of the complexes C_1^i is incident with any element of any of the others.

There are two non-zero elements in each column of H_1. Hence if we add the rows corresponding to any of the blocks I, II, etc. the sum is zero (mod. 2) in every column. Hence the rows of H_1 are connected by R_0 linear relations.

Any linear combination (mod. 2) of the rows of H_1 corresponds to adding a certain number of them together. If this gave zeros in all the columns it would mean that there were two or no 1's in each column of the matrix formed by the given rows, and this would mean that any 1-cell incident

with one of the 0-cells corresponding to these rows would also be incident with another such 0-cell. These 0-cells and the 1-cells incident with them would therefore form a sub-complex of C_1 which was not connected with any of the remaining 0-cells and 1-cells of C_1. Hence it would consist of one or more of the complexes C_1^i $(i = 1, 2, \cdots, R_0)$ and the linear relations with which we started would be dependent on the R_0 relations already found. Hence there are exactly R_0 linearly independent linear relations among the rows of H_1, so that if ρ_1 is the rank of H_1,

$$\rho_1 = \alpha_0 - R_0.$$

It follows that there is a set of $\alpha_0 - R_0$ columns of H_1 upon which all columns are linearly dependent. Since every column of H_1 is a solution of (H_0) and since all solutions of (H_0) are linearly dependent on $\alpha_0 - R_0$ linearly independent solutions, all solutions of (H_0) are linearly dependent on columns of H_1. In other words *any bounding 0-circuit is the sum of some of the 0-circuits which bound the 1-cells* $a_1^1, \cdots, a_{\alpha_1}^1$.

A linearly independent set of solutions of a set of linear equations upon which all other solutions are linearly dependent is called a *complete set* of solutions. Thus a set of ρ_1 linearly independent columns of H_1 forms a complete set of solutions of (H_0). The corresponding set of 0-circuits is also called a *complete set*.

21. If $R_0 = 1$ the complex C_1 is connected and all its 0-circuits are bounding and expressible linearly (mod. 2) in terms of $\alpha_0 - 1$ of the 0-circuits which bound 1-cells.

In case $R_0 > 1$, a 0-circuit obtained by taking two points, one from each of a pair of the sub-complexes C_1^i $(i = 1, 2, \cdots, R_0)$ is a non-bounding 0-circuit, while one obtained by taking two points from the same complex C_1^i is bounding.

If $R_0 = 2$ any two 0-cells are both in C_1^1, or both in C_1^2, or one in C_1^1 and the other in C_1^2. A pair of the last type forms a non-bounding 0-circuit and all non-bounding 0-circuits are of this type. If $a_i^0 a_k^0$ is a 0-circuit of the last type any

other non-bounding 0-circuit $a_l^0 a_m^0$ is such that one of its points, say a_l^0, is in the same connected complex with a_i^0 and the other with a_k^0. Hence $a_l^0 a_m^0$ is the sum (mod. 2) of $a_i^0 a_k^0$ and the two bounding 0-circuits $a_i^0 a_l^0$ and $a_k^0 a_m^0$. Hence any non-bounding 0-circuit is obtainable by adding bounding 0-circuits to a fixed non-bounding 0-circuit.

By a repetition of this reasoning one finds in the general case that $R_0 - 1$ is *the number of non-bounding 0-circuits which must be adjoined to the bounding ones in order to have a set in terms of which all the 0-circuits are linearly expressible (mod. 2).* These $R_0 - 1$ non-bounding 0-circuits can obviously be chosen to consist of the pairs of 0-cells. a_1^0, a_i^0 $(i = m_1 + 1, m_2 + 1, \cdots, m_{R_0 - 1} + 1)$.

One-dimensional Circuits

22. A connected linear graph each vertex of which is an end of two and only two 1-cells is called a *one-dimensional circuit* or *a 1-circuit.* By the theorems of § 5 any closed curve is decomposed by any finite set of points on it into a 1-circuit. Conversely, it is easy to see that the set of all points on a 1-circuit is a simple closed curve. It is obvious, further, that any linear graph such that each vertex is an end of two and only two 1-cells is either a 1-circuit or a set of 1-circuits no two of which have a point in common.

Consider a linear graph C_1 such that each vertex is an end of an even number of edges. Let us trace a path on C_1 starting at a 0-cell and not covering any 1-cell more than once. As a result of the hypothesis, we must eventually reach some 0-cell for the second time, hence have traced a 1-circuit. We remove this 1-circuit and replace the necessary 0-cells. Since the resulting complex has the property originally assumed for C_1, it follows that we can repeat the process till there is nothing left. Hence C_1 consists of a number of 1-circuits which have only a finite number of 0-cells in common.

It is obvious that a linear graph composed of a number of closed curves having only a finite number of points in common

has an even number of 1-cells incident with each vertex. Hence *a necessary and sufficient condition that C_1 consist of a number of 1-circuits having only 0-cells in common is that each 0-cell of C_1 be incident with an even number of 1-cells.* A set of 1-circuits having only 0-cells in common will be referred to briefly as a set of 1-circuits.

23. The sum of the symbols $(x_1, x_2, \cdots, x_{\alpha_0})$ for the 0-circuits which bound the 1-cells of a 1-circuit is $(0, 0, \cdots, 0)$ because each 0-cell appears in two and only two of these 0-circuits. Hence *any 1-circuit or set of 1-circuits determines a linear relation, modulo 2, among the bounding 0-circuits.*

Conversely, *any linear relation among the 0-circuits which bound 1-cells of a complex determines a 1-circuit or set of 1-circuits.* For if the sum of a set of 0-circuits reduces to $(0, 0, \cdots, 0)$ each 0-cell must enter in an even number of 0-circuits, i. e., as an end of an even number of 1-cells.

24. Let us now inquire under what circumstances a symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ for a one-dimensional complex contained in C_1 will represent a 1-circuit or a system of 1-circuits.

Consider the sum

$$\eta^1_{i1} x_1 + \eta^1_{i2} x_2 + \cdots + \eta^1_{i\alpha_1} x_{\alpha_1}$$

where the coefficients η^1_{ij} are the elements of the ith row of H_1. Each term $\eta^1_{ij} x_j$ of this sum is 0 if a^1_j is not in the set of 1-cells represented by $(x_1, x_2, \cdots. x_{\alpha_1})$ because in this case $x_j = 0$; it is also zero if a^1_j is not incident with a^0_i because $\eta^1_{ij} = 0$ in this case. The term $\eta^1_{ij} x_j = 1$ if a^1_j is incident with a^0_i and in the set represented by $(x_1, x_2, \cdots x_{\alpha_1})$ because in this case $\eta^1_{ij} = 1$ and $x_j = 1$. Hence there are as many non-zero terms in the sum as there are 1-cells represented by $(x_1, x_2, \cdots, x_{\alpha_1})$ which are incident with a^0_i. Hence by § 22 the required condition is that the number of non-zero terms in the sum must be even. In other words if the x's and η^1_{ij}'s are reduced modulo 2 as explained in § 14 we must have

$$(H_1) \qquad \sum_{j=1}^{\alpha_1} \eta^1_{ij} x_j = 0 \qquad (i = 1, 2, \cdots. \alpha_0)$$

if and only if $(x_1, x_2, \cdots, x_{\alpha_1})$ represents a 1-circuit or set of 1-circuits. The matrix of this set of equations (or congruences, mod. 2) is H_1.

25. If the rank of the matrix H_1 of the equations (H_1) be ϱ_1 the theory of linear homogeneous equations (congruences, mod. 2) tells us that there is a set of $\alpha_1 - \varrho_1$ linearly independent solutions of (H_1) upon which all other solutions are linearly dependent. This means geometrically that *there exists a set of $\alpha_1 - \varrho_1$ 1-circuits or systems of 1-circuits from which all others can be obtained by repeated applications of the operation of adding (mod. 2) described in* § 14. We shall call this a *complete set* of 1-circuits or systems of 1-circuits.

Since $\varrho_1 = \alpha_0 - R_0$ (§ 20), the number of solutions of (H_1) in a complete set is

$$\mu = \alpha_1 - \alpha_0 + R_0.$$

where μ is the cyclomatic number defined in § 13. For the sake of uniformity with a notation used later on we shall also denote μ by $R_1 - 1$. Thus we have

$$\alpha_0 - \alpha_1 = 1 + R_0 - R_1.$$

Trees

26. A connected linear graph which contains no 1-circuits is called a *tree*. As a corollary of the last section it follows that *a linear graph is a set of R_0 trees if and only if $\mu = 0$.*

Any connected linear graph C_1 can be reduced to a tree by removing μ properly chosen 1-cells. For let a_p^1 $(p = i_1, i_2, \cdots, i_{\varrho_1})$ be a set of 1-cells whose boundaries form a complete set of 0-circuits (§ 20). The remaining 1-cells of C_1 are μ in number and will be denoted by a_p^1 $(p = j_1, j_2, \cdots, j_\mu)$. If these μ 1-cells are removed from C_1 the linear graph T_1 which remains is connected because every bounding 0-circuit of C_1 is linearly expressible in terms of the boundaries of the 1-cells a_p^1 $(p = i_1, i_2, \cdots, i_{\varrho_1})$ of T_1 and hence any two 0-cells of C_1 are joined by a curve composed of 1-cells of T_1. But since the cyclomatic number of C_1 is $\mu = \alpha_1 - \alpha_0 + 1$,

the removal of μ 1-cells reduces it to 0 and hence reduces C_1 to a tree. In like manner, if C_1 is a linear graph for which $R_0 > 1$, it can be reduced to R_0 trees by removing $\mu = \alpha_1 - \alpha_0 + R_0$ properly chosen 1-cells.

27. There is at least one 1-circuit of C_1 which contains the 1-cell $a^1_{j_1}$, for otherwise C_1 would be separated into two complexes by removing this 1-cell. Call such a 1-circuit C^1_1. In the complex obtained by removing $a^1_{j_1}$ from C_1 there is, for the same reason, a 1-circuit C^2_1 which contains $a^1_{j_2}$, and so on. Thus there is a set of 1-circuits $C^1_1, C^2_1, \cdots, C^\mu_1$ such that C^p_1 $(p = 1, 2, \cdots, \mu)$ contains $a^1_{j_p}$. These 1-circuits are linearly independent because C^{k-1}_1 contains a 1-cell, $a^1_{j_{k-1}}$, which does not appear in any of the circuits $C^\mu_1, C^{\mu-1}_1, \cdots, C^k_1$ and therefore cannot be linearly dependent on them. Hence $C^1_1, C^2_1, \cdots, C^\mu_1$ constitute a complete set of 1-circuits. This sharpens the theorem of § 25 a little in that it establishes that there is a complete set of solutions of (H_1) each of which represents a single 1-circuit.

Geometric Interpretation of Matrix Products

28. According to the definition of multiplication of matrices,

$$\|a_{ij}\| \cdot \|b_{jk}\| = \|c_{ik}\|$$

if and only if

$$\sum_{j=1}^{\beta} a_{ij}\, b_{jk} = c_{ik},$$

β being the number of columns in $\|a_{ij}\|$ and the number of rows in $\|b_{jk}\|$.

Hence the equations (H_0) of § 19 are equivalent to the matrix equation,

$$H_0 \cdot \begin{Vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_0} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Vmatrix},$$

in which the matrix on the right has one column containing R_0 zeros.

Since each column of the matrix H_1 is the symbol (as defined in § 14) for a bounding 0-circuit, (i. e., the jth column is the symbol for the 0-circuit which bounds a_j^1) any column of H_1 is a solution $(x_1, x_2, \cdots, x_{\alpha_0})$ of the set of equations (H_0). By the remark above we may express this result in the form,

$$H_0 \cdot H_1 = 0.$$

where 0 is the symbol for a matrix all of whose elements are zero.

29. By the *boundary* of a one-dimensional complex is meant the set of 0-cells each of which is incident with an odd number of 1-cells of the complex. So, for example, a 1-circuit is a linear graph which has no boundary.

From the definition (§ 14) of addition (mod. 2) of sets of points it is clear that the sum of the boundaries of two 1-cells is the boundary of the complex consisting of the two 1-cells and their ends. By repeated application of this reasoning we prove that the boundary of any one-dimensional complex is an even number of 0-cells, i. e., a number of 0-circuits.

Now consider a one-dimensional complex C_1' represented by the symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ for its 1-cells. According to the reasoning in § 24 each term of

$$\eta_{i1}^1 x_1 + \eta_{i2}^1 x_2 + \cdots + \eta_{i\alpha_1}^1 x_{\alpha_1}$$

is 1 or 0 according as the corresponding 1-cell is or is not both in C_1' and incident with a_i^0. Hence this expression is 1 or 0 (mod. 2) according as a_i^0 is or is not a boundary point of C_1'. Hence if we set

$$\eta_{i1}^1 x_1 + \eta_{i2}^1 x_2 + \cdots + \eta_{i\alpha_1}^1 x_{\alpha_1} = y_i \qquad (i = 1, 2, \cdots, \alpha_0)$$

the symbol $(y_1, y_2, \cdots, y_{\alpha_0})$ thus determined represents the set of points which bounds C_1'.

Recalling the rule for multiplying matrices, we see that this result may be stated as follows:

$$H_1 \cdot \begin{Vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_1} \end{Vmatrix} = \begin{Vmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{\alpha_0} \end{Vmatrix}$$

if and only if $(y_1, y_2, \cdots, y_{\alpha_0})$ denotes the set of points which bounds the complex denoted by $(x_1, x_2, \cdots, x_{\alpha_1})$.

Reduction of H_0 and H_1 to Normal Form

30. Let us define two matrices B_0 and B_1 as follows:

B_0 is a matrix of α_0 rows and α_0 columns of which the first column is the symbol for a_0^1, the next $R_0 - 1$ columns are the symbols for the non-bounding 0-circuits enumerated at the end of § 21, and the last $\alpha_0 - R_0$ columns are the symbols for the boundaries of the 1-cells a_j^1 $(j = i_1, i_2, \cdots, i_{\varrho_1})$ of the trees of § 26.

B_1 is a matrix of α_1 rows and α_1 columns of which the first ϱ_1 columns are the symbols for a_j^1 $(j = i_1, i_2, \cdots, i_{\varrho_1})$, and the last $\alpha_1 - \varrho_1$ columns are the symbols for the 1-circuits $C_1^1, C_1^2, \cdots, C_1^{\mu}$.

The determinants of these two matrices are evidently 1 (mod. 2) because the columns of B_0 represent a linearly independent set of 0-dimensional complexes and the columns of B_1 a linearly independent set of 1-dimensional complexes.

The matrix B_0 has the properties: (1) all bounding 0-circuits are linearly dependent (mod. 2) upon the 0-circuits represented by its last ϱ_1 columns; (2) all non-bounding 0-circuits are linearly dependent on its last $\alpha_0 - 1$ columns; (3) all sets of 0-cells are linearly dependent on all its columns.

The matrix B_1 has the properties: (1) all 1-circuits are linearly dependent upon the 1-circuits represented by its last μ columns and (2) all sets of 1-cells are linearly dependent on all its columns.

31. From § 29 and the definition of B_1 it is clear that the first ϱ_1 columns of the product $H_1 \cdot B_1$ must be the symbols

for the boundaries of the 1-cells represented by the first ρ_1 columns of B_1. Hence the first ρ_1 columns of the product $H_1 \cdot B_1$ are the same as the last ρ_1 columns of B_0. The remaining columns of $H_1 \cdot B_1$ must be composed entirely of zeros since the remaining columns of B_1 represent 1-circuits. Hence

(1) $$H_1 \cdot B_1 = A_0 \cdot H_1^*,$$

where

$$H_1^* = \begin{Vmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{Vmatrix}$$

is a matrix of α_0 rows and α_1 columns of which all elements are 0's except the first ρ_1 elements of the main diagonal, and A_0 is a matrix of α_0 rows and α_0 columns whose first $\rho_1 = \alpha_0 - R_0$ columns are identical with the last ρ_1 columns of B_0 and whose last R_0 columns are identical with the first R_0 columns of B_0. Since the determinant of B_0 is 1, the determinant of A_0 is 1. Hence (1) may be written

(2) $$A_0^{-1} \cdot H_1 \cdot B_1 = H_1^*.$$

From the point of view of the algebra of matrices (mod. 2) the determination of the two matrices A_0^{-1} and B_1 is the solution of the problem of reducing H_1 to its normal or unitary form, H_1^*. Geometrically (cf. § 30) these matrices may be regarded as summarizing the theory of circuits in a linear graph. It will be found that this geometrical significance of the reduction of H_1 to its normal form generalizes to n dimensions. For the sake of completeness we shall also carry out the analogous reduction of H_0.

32. From § 28 and the definition of B_0 it is clear that

$$(1)\qquad H_0 \cdot B_0 = \left\|\begin{matrix} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{matrix}\right\| .$$

the right-hand member of this equation being a matrix of R_0 rows and α_0 columns. Each of the first R_0 columns of this matrix contains a 1 for each of the complexes C_1^i $(i = 1, 2, \cdots, R_0)$ which contains a 0-cell of the set represented by the corresponding columns of B_0. The last $\alpha_0 - R_0$ columns contain nothing but 0's because the last $\alpha_0 - R_0$ columns of B_0 represent bounding 0-circuits. This equation may also be written in the form

$$(2)\qquad H_0 \cdot B_0 = A \cdot H_0^*$$

in which A is a square matrix of R_0 columns identical with the first R_0 columns of $H_0 \cdot B_0$ and H_0^* is a matrix of R_0 rows and α_0 columns all elements of which are 0 except the R_0 elements of the main diagonal, which are all 1.

The determinant of the matrix A is unity and A therefore has a unique inverse A^{-1}. Hence (2) becomes

$$(3)\qquad A^{-1} \cdot H_0 \cdot B_0 = H_0^* .$$

Thus A^{-1} and B_0 are a pair of matrices by means of which H_0 is transformed to the normal form H_0^*.

Oriented Cells

33. We turn now to the notion of "orientation" or "sense of description" of a complex. The definitions adopted will doubtless seem very artificial, but this is bound to be the case in defining any idea so intuitionally elemental as that of "sense."

A 0-cell associated with the number $+1$ or -1 shall be called an *oriented 0-cell* or *oriented point.** In the first case

* In analytic applications the number ± 1 associated with a point is usually determined by the sign of a functional determinant.

the oriented 0-cell is said to be *positively oriented* and in the second case it is said to be *negatively oriented*; the two oriented points are called *negatives* of each other. A set of oriented 0-cells is called an *oriented 0-dimensional complex.*

A pair of oriented 0-cells, formed by associating one point of a 0-circuit with $+1$ and the other with -1 shall be called an *oriented 0-circuit* or an *oriented 0-dimensional manifold.* If a 0-circuit is bounding, ony oriented 0-circuit formed from it is also said to be *bounding.*

34. The ends a_1^0, a_2^0 of a 1-cell a^1 when associated each with $+1$ determine two oriented 0-cells which may be called σ_1^0 and σ_2^0 respectively. Therefore the ends of a^1 determine two oriented 0-circuits, namely σ_1^0, $-\sigma_2^0$ and $-\sigma_1^0$, σ_2^0. The object formed by associating a^1 with either of these 0-circuits is called an *oriented 1-cell.*

The oriented 1-cell σ^1 formed by associating a^1 with σ_1^0, $-\sigma_2^0$ is said to be *positively related* to σ_1^0 and $-\sigma_2^0$ and *negatively related* to $-\sigma_1^0$ and σ_2^0. An oriented 0-cell is said to be positively or negatively related to an oriented 1-cell according as the 1-cell is positively or negatively related to it.

The point a_1^0 is called the *terminal point* and a_2^0 the *initial point* of the oriented 1-cell σ^1 formed by associating a^1 with σ_1^0, $-\sigma_2^0$. In diagrams it is convenient to denote an oriented 1-cell by marking it with an arrow pointing from the initial point to the terminal point.

In the following sections we shall denote the oriented 0-cells obtained by associating each of the 0-cells $a_1^0, a_2^0, \cdots, a_{\alpha_0}^0$ of a complex C_1 with $+1$, by $\sigma_1^0, \sigma_2^0, \cdots, \sigma_{\alpha_0}^0$ respectively. We shall also denote an arbitrary one of the two oriented 1-cells which can be formed from a_i^1 $(i = 1, 2, \cdots, \alpha_1)$ by σ_i^1. Any set of oriented 1-cells will be called an *oriented one-dimensional complex.* Thus any linear graph can be converted into an oriented complex in 2^{α_1} ways.

35. The cells of a 1-circuit, when oriented by the process described above, give rise to a sequence of oriented 0-cells and 1-cells,

$$(1) \qquad \sigma_1^0,\ \sigma_1^1,\ \sigma_2^0,\ \sigma_2^1,\ \cdots \sigma_{\alpha_0}^0,\ \sigma_{\alpha_0}^1,\ \sigma_1^0,$$

in which each oriented cell is either positively or negatively related to the one which follows it. According to the convention that σ_i^0 is formed from a_i^0 by associating it with $+1$, each σ_i^1 is negatively related to the oriented 0-cell which follows it if it is positively related to the one which precedes it, and *vice versa.* Hence by assigning the notation so that σ_j^1 is in every case positively related to the oriented 0-cell which precedes it in the sequence (1) we can arrange that $\sigma_1^1, \sigma_2^1, \cdots, \sigma_{\alpha_1}^1$, represent a set of oriented 1-cells such that each oriented 0-cell positively related to one oriented 1-cell of the set is negatively related to another. Such an oriented complex formed from the 1-cells of a 1-circuit is called an *oriented 1-circuit.*

It is obvious that the only other oriented 1-circuit which can be formed from the given 1-circuit is that composed of $-\sigma_1^1$, $-\sigma_2^1$, $\cdots$, $-\sigma_{\alpha_0}^1$. For if one of the oriented 1-cells in an oriented 1-circuit be replaced by its negative each of the other 1-cells must be replaced by its negative. The other oriented complexes which can be formed from the 1-circuit are not oriented 1-circuits.

Intuitionally this discussion means that if the oriented 1-cells of an oriented 1-circuit are marked by arrows as in § 34, the arrows must all be pointed in the same direction.

Matrices of Orientation

36. The relations between the oriented 0-cells and oriented 1-cells, which can be formed from the cells of a complex C_1 may be studied by means of two matrices which are closely analogous to H_0 and H_1. The new matrices will be called *matrices of orientation*, and denoted by E_0 and E_1. In our treatment they are derived from H_0 and H_1 and their theory is entirely parallel to that of H_0 and H_1. They are, however, the one- and two-dimensional instances of the matrices E_i which form the central element in Poincaré's work on Analysis

Situs. The matrix E_1 may be said to date back to the article by G. Kirchoff in Poggendorf's Annalen der Physik, Vol. 72 (1847), p. 497, on the flow of electricity through a network of wires, in which Kirchoff made use of a system of linear equations having E_1 as its matrix. This paper is doubtless the first important contribution to the theory of linear graphs.

37. Any set of oriented 0-cells may be denoted by a symbol $(x_1, x_2, \cdots, x_{\alpha_0})$ in which x_i is $+1$ if σ_i^0 is in the set, -1 if $-\sigma_i^0$ is in the set, and 0 if neither σ_i^0 nor $-\sigma_i^0$ is in the set. The symbols for the bounding oriented 0-circuits of a complex C_1 satisfy a set of equations, (E_0), identical with the equations (H_0) of § 19 except that the variables are taken to be integers instead of being reduced modulo 2. The corresponding matrix will be denoted by

$$E_0 = \|\epsilon_{ij}^0\| \qquad (i = 1, 2, \cdots, R_0;\ j = 1, 2, \cdots, \alpha_0).$$

If the complex is connected, $R_0 = 1$ and this matrix reduces to a one-rowed matrix

$$\|1, 1, \cdots, 1\|$$

all of whose α_0 elements are unity. The equations (E_0) have $\alpha_0 - R_0$ linearly independent solutions, and if r_0 is the rank of E_0

$$r_0 = \rho_0 = R_0.$$

38. The relations between the oriented 0-cells σ_i^0 and oriented 1-cells σ_j^1 of an oriented complex C_1 may be denoted by a matrix

$$E_1 = \|\epsilon_{ij}^1\| \qquad (i = 1, 2, \cdots, \alpha_0;\ j = 1, 2, \cdots, \alpha_1)$$

in which ϵ_{ij}^1 is $+1$ if σ_i^0 is positively related to σ_j^1, is -1 if σ_i^0 is negatively related to σ_j^1, and is 0 if a_i^0 is not an end of a_j^1.

This matrix can be formed from H_1 by changing a 1 in each column to -1, for each σ_i^1 is positively related to one of the σ^0's formed from the ends of a_i^1 and negatively related to the other. The choice of the -1 is determined by the arbitrary choice in the definition of σ_i^1.

For example, the vertices and edges of the tetrahedron in Fig. 1 when oriented as indicated by the arrows constitute an oriented complex represented by the following matrix:

$$E_1 = \begin{Vmatrix} -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

39. Each column of the matrix E_1 is the symbol (§ 37) for a bounding oriented 0-circuit and hence is a solution of the set of equations (E_0). In the notation of matrices, this means

(1) $$E_0 \cdot E_1 = 0.$$

The matrix E_1 falls into a set of matrices I, II, III, etc. corresponding to those into which H_1 is decomposed in § 20. The sum of the rows of any one of these matrices I, II, III is zero because each column has one $+1$ and one -1. On the other hand the rows of such a matrix, say I, cannot be subject to any other linear relation because one of the variables could be eliminated between this relation and the one which states that the sum of the rows is zero, and the resulting relation, after its coefficients were divided by their H. C. F. and then reduced modulo 2, would give a linear relation among the rows of H_1 of a type which has been shown in § 20 to be non-existent. Hence the rows of E_1 are subject to R_0 linearly independent linear relations. Hence if r_1 denote the rank of E_1,

$$r_1 = \varrho_1 = \alpha_0 - R_0.$$

40. The form of the matrices E_0 and E_1 has been limited somewhat by the convention that $\sigma_1^0, \sigma_2^0, \cdots, \sigma_{\alpha_0}^0$ denote 0-cells each associated with $+1$. If we interchange the significance of σ_i^0 and $-\sigma_i^0$, so that σ_i^0 represents a_i^0 associated with -1, it is necessary to change the 1 in the ith column of E_0 to -1 and to make corresponding changes in the columns of E_1.

The rest of the discussion on this slightly more general foundation does not differ in essentials from that already given.

Oriented 1-Circuits

41. Every oriented 1-circuit corresponds to a linear relation among the oriented 0-circuits which bound the oriented 1-cells of which it is composed, for if a given oriented 0-cell is positively related to one such oriented 1-cell, its negative is, by the terms of the definition, positively related to another oriented 1-cell of the oriented 1-circuit. Conversely any linear relation among the bounding 0-circuits determines an oriented 1-circuit or set of oriented 1-circuits. All this is analogous to § 23. Taken with § 39 it establishes that the number of linearly independent linear relations among bounding oriented 0-circuits is the same as among bounding 0-circuits when reduced modulo 2.

42. Any set of oriented 1-cells of a complex C_1 may be denoted by $(x_1, x_2, \cdots, x_{\alpha_1})$ where $x_i = 1$ if σ_i^1 is in the set, $x_i = -1$ if $-\sigma_i^1$ is in the set, and $x_i = 0$ if neither σ_i^1 nor $-\sigma_i^1$ is in it. A necessary and sufficient condition that such a symbol represent an oriented 1-circuit or set of oriented 1-circuits is that it satisfy the system of equations,

$$(E_1) \qquad \sum_{j=1}^{\alpha_1} \epsilon_{ij}^1 x_j = 0 \qquad (i = 1, 2, \cdots, \alpha_0),$$

the matrix of which is E_1. For in this set, the equation,

$$(1) \qquad \epsilon_{i1}^1 x_1 + \epsilon_{i2}^1 x_2 + \cdots + \epsilon_{i\alpha_1}^1 x_{\alpha_1} = 0$$

corresponds to the oriented 0-cell σ_i^0. A term $\epsilon_{ij}^1 x_j$ of the left member is zero if $\epsilon_{ij}^1 = 0$ or if $x_j = 0$, that is, if σ_i^1 is not an end of σ_j^1 or if the set of oriented 1-cells does not contain $\pm\sigma_j^1$. The term $\epsilon_{ij}^1 x_j$ is $+1$ if ϵ_{ij}^1 and x_j are of the same sign, that is if the set of oriented 1-cells contains σ_j^1 and the latter is positively related to σ_i^0 or if it contains $-\sigma_j^1$ and $-\sigma_j^1$ is positively related to σ_i^0; hence there are as many $+1$ terms in the left member of (1) as there are oriented 1-cells

in the set $(x_1 . x_2, \cdots . x_{\alpha_1})$ which are positively related to σ_i^0. In like manner there are as many -1 terms as there are oriented 1-cells in the set which are negatively related to σ_i^0. Hence *the left-hand member of* (1) *is the difference between the number of oriented* 1-*cells in the set which are positively related to* σ_i^0 *and the number which are negatively related to* σ_i^0. Hence an oriented 1-circuit satisfies the equations (E_1), and any solution of (E_1) of the kind in question must represent an oriented 1-circuit or a set of oriented 1-circuits.

Since the number of variables x_j in the equations (E_1) is α_1 and the rank of the matrix of coefficients is $\alpha_0 - R_0$ (cf. § 39) the number of solutions in a set on which all others are linearly dependent is μ where

$$\mu = \alpha_1 - \alpha_0 + R_0.$$

Such a set is obviously obtained by converting the μ 1-circuits of § 27 into oriented 1-circuits. The symbols $(x_1, x_2, \cdots, x_{\alpha_1})$ for these 1-circuits are linearly independent solutions of (E_1) in which the x's are 0 or ± 1.

It is obvious that the equations (E_1) have solutions in which the x's are integers different from 0 and ± 1. In order to interpret these solutions we shall return to the notion of a singular complex on C_1 (§ 8).

Symbols for Oriented Complexes

43. If a 0-cell a^0 on C_1 (in the sense of § 7) is associated with $+1$ or -1 the resulting oriented 0-cell σ^0 is said to be *on* C_1, and if a^0 coincides with a 0-cell a_i^0 of C_1, σ^0 is said to *coincide* with σ_i^0 or $-\sigma_i^0$ according as σ^0 is positively or negatively oriented.

Let C_1' be any linear graph on C_1 such that each 1-cell of C_1' covers a 1-cell of C_1 just once (cf. § 9). If the cells of both complexes are oriented, an oriented 1-cell σ_p^1 of C_1' will be said to *coincide* with an oriented 1-cell σ_q^1 of C_1 if and only if (1) each point of σ_p^1 coincides with a point of σ_q^1 and (2) each oriented 0-cell of C_1' is positively or negatively

related to σ^1_p according as it coincides with an oriented 0-cell of C_1 which is positively or negatively related to σ^1_q.

44. A symbol $(x_1, x_2, \cdots, x_{\alpha_i})$ in which the x's are positive or negative integers or 0 will be taken to represent a set of oriented i-cells ($i = 0$ or 1) on C_1 in which (1) if $x_j (j = 1, 2, \cdots, \alpha_i)$ is positive there are x_j oriented i-cells coinciding with σ^i_j, (2) if x_j is negative there are $-x_j$ oriented i-cells coinciding with $-\sigma^i_j$, and (3) if $x_j = 0$ there are no oriented i-cells coinciding with σ^i_j or $-\sigma^i_j$.

The object obtained by assigning orientations to the 1-cells of a complex is called an *oriented 1-dimensional complex*. A *singular oriented complex* is defined in similar manner. A singular oriented complex whose cells coincide with cells of C_1 determines a symbol $(x_1, x_2, \cdots, x_{\alpha_1})$. Conversely, any such symbol determines at least one oriented complex having that symbol for its oriented 1-cells. This complex can in general be constructed in a variety of ways, depending on how we join the 1-cells by 0-cells.

In case the numbers $x_j (j = 1, 2, \cdots, \alpha_i;\ i = 0, 1)$, have a common factor different from unity, i. e., in case

$$(x_1, x_2, \cdots, x_{\alpha_i}) = (z_1 d, z_2 d, \cdots, z_{\alpha_i} d),$$

any oriented complex whose symbol is $(z_1, z_2, \cdots, z_{\alpha_i})$ is said to be *covered d times* by a complex with symbol $(x_1, x_2, \cdots, x_{\alpha_i})$ formed by orienting the cells of a complex covering $(z_1, z_2, \cdots, z_{\alpha_i})$ d times in the sense of § 9.

45. If $(x_1, x_2, \cdots, x_{\alpha_i})$ and $(y_1, y_2, \cdots, y_{\alpha_i})$ are symbols for two sets of oriented i-cells ($i = 0, 1$), the symbol $(x_1 + y_1, x_2 + y_2, \cdots, x_{\alpha_i} + y_{\alpha_i})$ is called the *sum* of the two symbols and the set of oriented i-cells which it represents is called the *sum* of the two sets of oriented i-cells.

Given two (singular) oriented complexes, any oriented complex whose symbol is the sum of the symbols of the given complexes is called a *sum* of those complexes.

For example, in Fig. 1 the oriented 1-circuit composed of $\sigma^1_4, \sigma^1_5, \sigma^1_6$ may be denoted by (0, 0, 0, 1, 1, 1) and the oriented 1-circuit composed of $\sigma^1_2, \sigma^1_4, -\sigma^1_3$ may be denoted by (0, 1,

$-1, 1, 0, 0)$. Their sum is $(0, 1, -1, 2, 1, 1)$. If each of σ_2^1, σ_4^1 and $-\sigma_8^1$ be replaced by its negative the sum becomes $(0, -1, 1, 0, 1, 1)$. In the first case the sum determines a pair of oriented 1-circuits, σ_4^1 appearing once in each; in the second case the sum determines a single oriented 1-circuit.

It can be proved by an argument analogous to that used in § 22 that any solution of the equations (E_1) represents a set of oriented 1-circuits, two ore more of which may have a given oriented 1-cell in common.

46. By the *boundary* of an oriented 1-cell is meant the pair of oriented points which are positively related to it. By the *boundary* of any oriented one-dimensional complex is meant the sum of the boundaries of the oriented 1-cells composing it.

From this definition it follows directly that an oriented 1-circuit has no boundary and that any set of oriented 1-cells without a boundary may be regarded as a set of 1-circuits.

If $(x_1, x_2, \cdots, x_{\alpha_1})$ is the symbol for a single oriented 1-cell, it is obvious from the reasoning used in § 42 that $(y_1, y_2, \cdots, y_{\alpha_0})$ is the symbol for its boundary if and only if

$$(1) \qquad E_1 \cdot \begin{Vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_1} \end{Vmatrix} = \begin{Vmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{\alpha_0} \end{Vmatrix}.$$

But the most general symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ in which the x's are integers or zero can be expressed as a sum of symbols for oriented 1-cells, and by the algebraic properties of matrices.

$$(2) \qquad E_1 \cdot \begin{Vmatrix} x_1 + x_1' \\ x_2 + x_2' \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_1} + x_{\alpha_1}' \end{Vmatrix} = E_1 \cdot \begin{Vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_1} \end{Vmatrix} + E_1 \cdot \begin{Vmatrix} x_1' \\ x_2' \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_1}' \end{Vmatrix}.$$

Hence in the general case, $(y_1, y_2, \cdots, y_{\alpha_0})$ is the symbol for the boundary of $(x_1, x_2, \cdots, x_{\alpha_1})$ if and only if (1) is satisfied.

Normal Form for E_0

47. All columns, except the first one, of the matrix B_0 which appeared (§ 32) in the reduction of H_0 to normal form are symbols for 0-circuits. Hence by changing one of the 1's in each column after the first column to -1, B_0 is converted into a matrix, D_0, of which the first column represents the oriented 0-cell σ_1^0, the next R_0-1 columns represent linearly independent non-bounding oriented 0-circuits, and the last α_0-R_0 columns represent linearly independent bounding oriented 0-circuits. The product $E_0\cdot D_0$ is clearly obtained from $H_0\cdot B_0$ by changing one 1 to -1 in each column from the second to the R_0th. Hence

(1) $$E_0\cdot D_0 = C\cdot E_0',$$

where E_0' is the same as H_0' and C is obtained from A by changing one 1 into -1 in each column except the first. The determinant of C is ± 1. Hence there exists a matrix C^{-1} whose elements are integers and (1) can be written in the form

(2) $$C^{-1}\cdot E_0\cdot D_0 = E_0'.$$

The reduction of E_0 to normal form, therefore, is completely parallel to the corresponding reduction of H_0.

Matrices of Integers

48. The reduction of E_1 to normal form can be obtained directly from the general theory of matrices whose elements are integers.* The fundamental theorem of this theory is that for any matrix E of α_1 rows and α_2 columns whose elements are integers there exist two square matrices C and

* The part of this theory which is needed for our purposes is the subject of an expository article (Appendix II) by P. Franklin and the author in the Annals of Mathematics, Vol. 23 (1921), pp. 1–15.

D of α_1 rows and α_2 rows respectively, each of determinant ± 1, such that

$$(1) \qquad C \cdot E \cdot D = E^*$$

where E^* is a matrix of α_1 rows and α_2 columns

$$E^* = \left\| \begin{array}{cccccc} d_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ 0 & 0 & \cdots & d_r & \cdots & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{array} \right\|$$

in which d_1 is the highest common factor of all the elements of E, $d_1 d_2$ the H.C.F. of all the two-rowed determinants which can be found by removing rows and columns from E, and finally, $d_1 d_2 \cdots d_r$ the H.C.F. of all the r-rowed determinants which can be formed from E. The number d_1 is the H.C.F. of all the numbers $d_1 d_2 d_3 \cdots d_r$, d_2 is the H.C.F. of $d_2, d_3, \cdots, d_r$, etc.

The numbers $d_1, d_1, \cdots, d_r$ are called the *invariant factors*, or the *elementary divisors* of the matrix E. They are invariants in the sense that if E is multiplied on the left by a square matrix of α_1 rows and determinant ± 1 and on the right by any square matrix of α_2 rows and determinant ± 1, the resulting matrix will be such that the H.C.F. of all the k-rowed determinants which can be formed from it is $d_1 \cdot d_2 \cdot \cdots, d_k$ $(k = 1, 2, \cdots, r)$.

If all elements be reduced modulo 2, E reduces to a matrix H all of whose elements are 0 or 1. The equation (1) reduces to an equation like (2) of § 31. The rank of E differs from the rank of H by the number of d's which contain 2 as a factor.

Normal Form for E_1

49. Suppose we apply the theory just described to the reduction to normal form of the matrix E_1, with the reduction represented by the equation

(1) $$C_0^{-1} \cdot E_1 \cdot D_1 = E_1^*.$$

From this we get the following result: There exists a *complete set* of sets of 1-circuits, that is, one such that an arbitrary set of 1-circuits is a linear combination, with integral coefficients, of its members. To prove this, we first obtain from (1) the relation,

(2) $$E_1 \cdot D_1 = C_0 \cdot E_1^*.$$

Since the last $\alpha_1 - r_1$ columns of E_1^* are composed of zeros, the last $\alpha_1 - r_1$ columns of D_1 represent sets of 1-circuits, which must be independent, since the determinant of D_1 is not zero. Since the determinant of D_1 is actually ± 1, an arbitrary set of 1-circuits must be uniquely expressible as a linear combination, with integral coefficients, of the complexes represented by the columns of D_1. If this linear combination actually involved any of the first r_1 columns, the combination of the latter columns in question must represent a set of 1-circuits, and also be independent of the last $\alpha_1 - r_1$ columns, as all the columns of D_1 are linearly independent. Hence the equations (E_1) would have more than $\alpha_1 - r_1$ linearly independent solutions, which we know is not the case. Therefore the linear combination in question cannot involve any of the first r_1 columns, and we conclude that the last $\alpha_1 - r_1$ columns represent a complete set of sets of 1-circuits.

50. We shall now outline a proof of the fact that the invariant factors of E_1 are all $+1$. Suppose a certain invariant factor, say the jth, had a value greater than 1, say d. Then from (2) it follows that the jth column of C_0 would represent a set of 0-circuits which, taken d times, would bound a 1-dimensional complex represented by the jth column of D_1. When we go into the subject more deeply, in the general case (Chap. IV, § 30), we shall prove that under these conditions the set of 0-circuits in question could not bound when taken a smaller positive number of times than d. But it is not difficult to prove that if a set of

0-circuits bounds when taken d times, d positive, then it also bounds when taken once. Hence the existence of the invariant factor d would lead us to a contradiction; and it follows that all the invariant factors of E_1 are 1.

51. In view of the general theory it is seen that the matrix E_1 for a linear graph is characterized by the fact that *its invariant factors are all* $+1$. On this account the theory of the matrix E_1 is essentially the same as that of H_1. When we come to the generalizations to two and more dimensions, the invariant factors of the matrix will no longer have this simple property and the invariant factors will turn out to be important Analysis Situs invariants.

CHAPTER II

TWO-DIMENSIONAL COMPLEXES AND MANIFOLDS

Fundamental Definitions

1. In a Euclidean space three non-collinear points and the segments which join them by pairs constitute the boundary of a finite region in the plane of the three points. This region is called a *triangular region* or *two-dimensional simplex* and the three given points are called its *vertices*. The points of the boundary are not regarded as points of the region.

Consider any set of objects in (1–1) correspondence with the points of a two-dimensional simplex and its boundary. The objects corresponding to the points of the simplex constitute what is called a *two-dimensional cell* or *2-cell,* and those corresponding to the boundary of the simplex what is called the *boundary of the 2-cell.*

The objects which constitute a cell and its boundary will hereafter be referred to as "points," and the remarks in § 2, Chap. I, with regard to order relations are carried over without change to the two-dimensional case. The boundary of a 2-cell obviously satisfies the definition given in Chap. I of a closed curve.

2. A *two-dimensional complex* may be defined as a one-dimensional complex C_1 together with a number, α_2, of 2-cells whose boundaries are 1-circuits of the one-dimensional complex, such that each 1-cell is on the boundary of at least one 2-cell and no 2-cell has a point in common with another 2-cell or with C_1. The order relations of the points of the boundary of each 2-cell must coincide with the order relations determined among these points as points of the 1-circuit of the one-dimensional complex which coincides with the boundary. (Compare the footnote to § 2, Chap. III.)

The surface of a tetrahedron (cf. Fig. 1) is a simple example of a two-dimensional complex. Any polyhedron or combination of polyhedra in a Euclidean space will furnish a more complicated example.

An arbitrary subset of the 0-cells, 1-cells, and 2-cells of a two-dimensional complex will be occasionally referred to as a *generalized two-dimensional complex.*

3. The definitions of limit point and continuous transformation given in Chap. I may be generalized directly to two-dimensional complexes and we take them for granted without further discussion. As in § 4, Chap. I, two complexes are said to be *homeomorphic* if there exists a (1–1) continuous correspondence between them; and any such correspondence is called a *homeomorphism.* The two complexes will in general be defined in quite different ways so that the numbers α_0, α_1, α_2 are different; but if the two complexes are homeomorphic there is a (1–1) continuous correspondence between them as sets of points.

Any proposition about a complex or set of complexes which is unaltered under the group of all homeomorphisms of these complexes is called a proposition of *two-dimensional Analysis Situs.*

Matrices of Incidence

4. The 0-cells and 1-cells on the boundary of a 2-cell are said to be *incident* with the 2-cell and the 2-cell to be *incident* with the 0-cells and 1-cells of its boundary. The incidence relations between the 1-cells and 2-cells of a two-dimensional complex C_2 may be indicated by a table or matrix analogous to that described in § 17, Chap. I. The 2-cells, α_2 in number, shall be denoted by $a_1^2, a_2^2, \cdots, a_{\alpha_2}^2$. The matrix $H_2 = \|\eta_{ij}^2\|$ which describes the incidence relations between the 1-cells and 2-cells is such that $\eta_{ij}^2 = 0$ if a_i^1 is not incident with a_j^2 and $\eta_{ij}^2 = 1$ if a_i^1 is incident with a_j^2.

In the case of the tetrahedron in Fig. 1, let us denote the 2-cells opposite the vertices a_1^0, a_2^0, a_3^0, a_4^0 by a_1^2, a_2^2, a_3^2, a_4^2 respectively. The table of incidence relations becomes

	a_1^2	a_2^2	a_3^2	a_4^2
a_1^1	0	1	1	0
a_2^1	1	0	1	0
a_3^1	1	1	0	0
a_4^1	1	0	0	1
a_5^1	0	1	0	1
a_6^1	0	0	1	1

5. Since each column of H_2 contains α_1 elements it may be regarded as a symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ in the sense of § 15, Chap. I for a set of 1-cells. The jth column of H_2 is, in fact the symbol for the 1-cells on the boundary of the 2-cell a_j^2. It is therefore the symbol for a 1-circuit. Hence the *columns of H_2 are solutions of the equations* (H_1). That is to say

$$\sum_{j=1}^{\alpha_1} \eta_{ij}^1 \eta_{jk}^2 = 0 \qquad (i = 1, \cdots, \alpha_0, \; k = 1, \cdots, \alpha_2)$$

or, in terms of the multiplication of matrices,

$$(1) \qquad H_1 \cdot H_2 = 0,$$

where 0 stands for the matrix all of whose elements are zero.

It should be recalled here that we have already proved in § 28, Chap. I that

$$H_0 \cdot H_1 = 0.$$

The ranks of the matrices H_0, H_1, H_2, computed modulo 2, will be denoted by ϱ_0, ϱ_1, ϱ_2 respectively.

6. From the point of view of Analysis Situs a two-dimensional complex is fully described by the three matrices $H_0 . H_1, H_2$ for there is no difficulty in proving that if two two-dimensional complexes have the same matrices there is a (1–1) continuous correspondence between them. Our definitions are such that the boundary of every 1-cell is a pair of distinct points and the boundary of every 2-cell a non-singular curve. Hence a figure composed of a 1-cell incident with a 0-cell or a 2-cell is in (1–1) continuous correspondence with any other such figure.

If two complexes C_2 and $\bar{C}_2$ have the same matrices their 0-cells, 1-cells and 2-cells may be denoted by a_i^0, a_j^1, a_k^2 and b_i^0, b_j^1, b_k^2 in such a way that whenever a_j^1 for any values of i, j, k is incident with a_i^0 or a_k^2, the b_j^1 for the same value of j is incident with the b_i^0 or b_k^2 with the same value of i or k. A (1–1) continuous correspondence is then set up between C_2 and $\bar{C}_2$ by requiring: (1) that a_i^0 correspond to b_i^0 for each value of i, (2) that a_j^1 and its ends correspond to b_j^1 and its ends for each value of j in a (1–1) continuous correspondence such that the correspondence between the ends is that set up under (1), and (3) that a_k^2 and its boundary correspond to b_k^2 and its boundary in a (1–1) continuous correspondence by which the boundaries correspond in the correspondence set up under (2).

Subdivision of 2-Cells

7. The properties of a two-dimensional complex will be obtained by studying the combinatorial relations codified in the matrices H_0, H_1, H_2 in connection with the continuity properties of the 2-cell. The latter properties, according to the definition in § 1, depend on the order relations in a Euclidean plane and, in particular, on the theory of planar polygons. The theory of polygons can be built up in terms of the incidence matrices. For consider a set of n straight lines in a Euclidean plane. They separate it into a number α_2 of planar convex regions and intersect in a number α_0 of points which divide the lines into a number α_1 of linear convex regions. The α_0 points can be treated as 0-cells, the α_1 linear convex regions as 1-cells and the α_2 planar convex regions as 2-cells. Any polygon is a 1-circuit, and the theory of linear dependence as developed in our first chapter can be applied to the proof of the fundamental theorems on polygons. For the details of this theory, which belongs to affine geometry rather than to Analysis Situs, the reader is referred to Chapters II and IX of the second volume of Veblen and Young's Projective Geometry.

8. The (1–1) correspondence with the interior and boundary of a triangle which defines a 2-cell and its boundary determines a system of 1-cells in the 2-cell which are the correspondents of the straight 1-cells in the interior of the triangle. By regarding this system of 1-cells as the *straight 1-cells* and defining the *distance* between any two points of the 2-cell and its boundary as the distance between the corresponding two points of the interior of the triangle, we can carry over all the theorems of the elementary geometry of a triangle to the 2-cell. The notions of distance and straightness so developed, however, are not invariant under the group of homeomorphisms, and the corresponding theorems are not theorems of Analysis Situs. For purposes of Analysis Situs the theorem of interest here is simply that there exists a system of 1-cells which are in (1–1) continuous correspondence with the straight 1-cells of the interior of a triangle of the Euclidean plane.

Given two circles in a Euclidean plane, it is a simple matter to put them and their interiors in (1–1) continuous correspondence in such a way that the correspondence thus set up between the circles is any preassigned homeomorphism. It follows that if definitions of distance are assigned arbitrarily along all the 1-cells of any complex C_2, definitions of distance and straightness can then be assigned to the 2-cells and their boundaries in such a way that the distances thus assigned along the 1-cells agree with those previously assigned. If a 2-cell has only two 1-cells on its boundary, they are necessarily curved under the definitions of distance and straightness for that 2-cell. Otherwise they can be taken straight.

9. The following theorems follow immediately from the homeomorphism between a 2-cell and the triangle used in defining it:

If two points A and B of the boundary of a 2-cell a^2 are joined by a straight 1-cell a^1 consisting of points of a^2, the remaining points of a^2 constitute two 2-cells each of which is bounded by a^1, A, B and one of the two 1-cells into which the boundary of a^2 is divided by A and B.

If the boundaries of two 2-cells a_1^2 and a_2^2 have a 1-cell a^1 and its ends in common, and the 2-cells and their boundaries have no other common points, then a_1, a_1^2 and a_2^2 constitute a 2-cell.

If there is a (1–1) continuous correspondence F' between the boundaries of two 2-cells a_1^2 and a_2^2, there exists a (1–1) continuous correspondence F between the interior and boundary of a_1^2 and the interior and boundary of a_2^2 which effects the correspondence F' between the boundaries.

A point of a 2-cell can be joined to a set of points A_1, $A_2, \cdots, A_n$ of its boundary by a set of 1-cells $a_1^1, a_2^1. \cdots, a_n^1$ which are in the 2-cell and have no points in common. The 2-cell is thus decomposed into n 2-cells $a_1^2, a_2^2, \cdots, a_n^2$ such that the sum of their boundaries (mod. 2) is the boundary of a^2 and such that the incidence relations between them and $a_1^1, a_2^1, \cdots, a_n^1$ are the same as the incidence relations between the 0-cells and 1-cells of a 1-circuit.

Conversely, if $a_1^1, a_2^1, \cdots, a_n^1$ and $a_1^2, a_2^2, \cdots, a_n^2$ are 1-cells and 2-cells all incident with the same point a^0 and also incident with one another in such a way that the incidence relations between the 1-cells and 2-cells are the same as those between the 0-cells and 1-cells of a 1-circuit, and $a_1^1, a_2^1, \cdots, a_n^1$ are the only 1-cells that the boundaries of any two of the 2-cells $a_1^2, a_2^2, \cdots, a_n^2$ have in common, then the point a^0 and the points of $a_1^1, a_2^1, \cdots, a_n^1$ and $a_1^2, a_2^2, \cdots, a_n^2$ constitute a 2-cell a^2 which is bounded by the sum (mod. 2) of the boundaries of the 2-cells $a_1^2, a_2^2, \cdots, a_n^2$.

10. The first of the theorems in the last section is a special case of the theorem that any 1-cell which is in a 2-cell and joins two points of its boundary decomposes the 2-cell into two 2-cells. This more general theorem depends on the theorem of Jordan, that any simple closed curve in a Euclidean plane separates the plane into two regions, the interior and the exterior; and also on the theorem of Schoenflies that the interior of a simple closed curve is a 2-cell of which the curve is the boundary.

We shall not need to use these more general forms of the separation theorems because we need, in general, merely the

existence of curves which separate cells, and this is provided for in the theorems of the last section. In connection with the Jordan theorem, reference may be made to the proof by J. W. Alexander, Annals of Mathematics. Vol. 21 (1920), p. 180.

Maps

11. With the aid of the theorems on separation a 2-cell a^2 may be subdivided into further 2-cells as follows: Let any two points a_1^0 and a_2^0 of the boundary of the 2-cell be joined by a straight 1-cell a_1^1 consisting entirely of points of the 2-cell. The 2-cell is thus separated into two 2-cells a_1^2 and a_2^2. The boundary of a^2 is likewise separated into two 1-cells a_2^1 and a_3^1 which have a_1^0 and a_2^0 as ends. The 0-cells, 1-cells and 2-cells into which a^2 is thus subdivided constitute a 2-dimensional complex C_2 whose matrices are

$$H_0 = \left\| 1 \quad 1 \right\|, \quad H_1 = \left\| \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \right\|, \quad H_2 = \left\| \begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right\|.$$

The numbers α_0, α_1, α_2 for C_2 are respectively 2, 3, 2, so that

$$\alpha_0 - \alpha_1 + \alpha_2 = 1.$$

This subdivision of a^2 may be continued by two processes: (1) introducing a point of a 1-cell as a new 0-cell and (2) joining two 0-cells of the boundary of a 2-cell by a 1-cell composed entirely of points of the 2-cell. The first process increases the numbers of 0-cells and 1-cells each by 1. The second process increases the numbers of 1-cells and 2-cells each by 1. Hence any number of repetitions of the two processes leave the number $\alpha_0 - \alpha_1 + \alpha_2$ invariant.

Any two-dimensional complex obtainable from a 2-cell by subdivision of the kind described above is called a *simply connected map*; and it can easily be proved that any two-dimensional complex which is homeomorphic with the interior and boundary of a 2-cell is a simply connected map.

The number $\alpha_0 - \alpha_1 + \alpha_2$ determined by any complex C_2 having α_0 0-cells, α_1 1-cells and α_2 2-cells is called the *character-*

istic of C_2. Thus we have proved that *the characteristic of a simply connected map is* 1.

12. There are a number of interesting theorems about simply connected maps which must be omitted here because they are of too special a nature. Many of them are related to the *four color problem*: is it possible to color the cells of a simply connected map with four colors in such a way that no two 2-cells which are incident with the same 1-cell are colored alike? This problem is still unsolved, in spite of numerous attempts. In addition to the references in the Encyclopädie, Vol. III_1, p. 177, the following references may be cited: Birkhoff, The reducibility of maps, American Journal of Mathematics, Vol. 35, p. 115; Veblen, Annals of Mathematics, Vol. 14 (1912), p. 86; and an article by P. Franklin in the American Journal, Vol. 44 (1922), pp. 225–236.

Regular Subdivision

13. It will often be found convenient to work with complexes whose 2-cells are each incident with three 0-cells and three 1-cells. Such 2-cells will be called *triangles* and a complex subdivided into triangles will be said to be *triangulated.* Any complex C_2 may be triangulated by the following process which is called a *regular subdivision.*

Let $P_k^2\,(k = 1, 2, \cdots, \alpha_2)$ be an arbitrary point of the 2-cell a_k^2, $P_j^1\,(j = 1, 2, \cdots, \alpha_1)$ an arbitrary point of the 1-cell a_j^1 and $P_i^0\,(i = 1, 2, \cdots, \alpha_0)$ another name for the 0-cell a_i^0. The points $P_j^i\,(i = 0, 1, 2;\ j = 1, 2, \cdots, \alpha_i)$ are to be the vertices of the complex $\overline{C}_2$.

Each P_j^1 separates the a_j^1 on which it lies into two 1-cells. The 1-cells so defined are to be among the 1-cells of $\overline{C}_2$. The remaining 1-cells of $\overline{C}_2$ are obtained by joining each P_k^2 to each of the points P_i^0 and P_j^1 of the boundary of a_k^2 by a straight 1-cell in a_k^2. Each 2-cell a_k^2 is thus decomposed into a set of 2-cells each of which is bounded by three of the 1-cells of $\overline{C}_2$, one on the boundary of a_k^2 and two interior to a_k^2. The 2-cells thus obtained are the 2-cells of $\overline{C}_2$.

The complex $\bar{C}_2$ is called a *regular subdivision* of C_2 and is also called a *regular complex*. No two 0-cells of $\bar{C}_2$ are joined by more than one 1-cell of $\bar{C}_2$. Moreover no 1-cell of $\bar{C}_2$ joins two points P^i_k, P^i_l which have equal superscripts. Hence any 1-cell of $\bar{C}_2$ may be denoted by $P^i_k P^j_l$ with $i<j$.

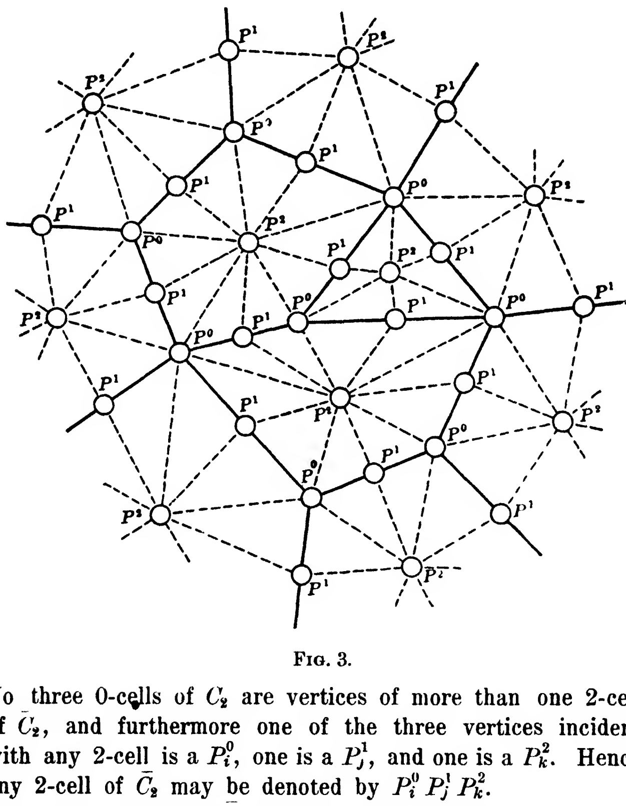

Fig. 3.

No three 0-cells of $\bar{C}_2$ are vertices of more than one 2-cell of $\bar{C}_2$, and furthermore one of the three vertices incident with any 2-cell is a P^0_i, one is a P^1_j, and one is a P^2_k. Hence any 2-cell of $\bar{C}_2$ may be denoted by $P^0_i P^1_j P^2_k$.

14. Any vertex of $\bar{C}_2$ together with the 1-cells and 2-cells which are incident with it is called a *triangle star*, and the vertex is called the *center* of the triangle star. Any point P of C_2 may be taken as the center of a triangle star of $\bar{C}_2$. For if P is on a 1-cell a^1_i of C_2 it can be chosen as the corre-

sponding P_i^1 and if it is on a 2-cell a_i^2 it can be chosen as the corresponding P_i^2. The set of all triangle stars of a given regular complex is such that each point of the complex is in at least one of them.

If C_2 is itself regular any two vertices of C_2 which are within or on the boundary of a triangle star of $\bar{C}_2$ are joined by a 1-cell of C_2.

15. The method of regular subdivision is useful in continuity arguments where it is desirable to subdivide a given complex into "arbitrarily small" cells. Let a complex C_2 in which a definition of straight lines and of distance has been introduced as described in § 8, be subjected to a regular subdivision into a complex C_2^1 and let C_2^1 be regularly subdivided into C_2^2, and so on, thus determining a sequence of complexes C_2, $C_2^1, \cdots, C_2^n, \cdots$, each of which is a regular subdivision of the one preceding it. Let us require also that each new 0-cell introduced in a 1-cell in the process of subdivision shall be the mid-point of the 1-cell, that each point interior to a triangular 2-cell (the point P_k^2 of § 13) shall be the center of gravity (intersection point of the medians) of the triangle, and that the 1-cells introduced shall be straight. With these conventions, it is evident that for every number $\delta > 0$ there exists a number N_δ such that if $n > N_\delta$ every 1-cell in C_2^n is of length less than δ.

16. The relationship between C_2 and C_2 may be stated as follows: (1) each 2-cell a_k^2 of C_2 is composed of P_k^2 and all the 1-cells $P_i^0 P_k^2$ or $P_j^1 P_k^2$ and all 2-cells $P_i^0 P_j^1 P_k^2$, of $\bar{C}_2$ incident with P_k^2; (2) each 1-cell a_j^1 of C_2 is composed of P_j^1 and the two 1-cells $P_i^0 P_j^1$ of C_2 incident with P_j^1; and (3) each 0-cell a_i^0 of C_2 is the vertex P_i^0 of $\bar{C}_2$.

Hence the complex $\bar{C}_2$ may be converted into C_2 by a series of steps of two sorts; (1) combine two 2-cells whose boundaries have one and only one 1-cell in common into a new 2-cell, suppressing the common 1-cell and (2) combine two 1-cells both incident with a 0-cell which is not incident with any other 1-cell into a new 1-cell, suppressing the common 0-cell.

The first type of step requires that the matrix H_2 of C_2 be modified by adding the column representing one of the two 2-cells to the one representing the other, removing the column representing the first of the two 2-cells, and also removing the row corresponding to the 1-cell which is suppressed. The row which is removed contained only two 1's before the two columns were added, because the 1-cell to which it corresponds is incident with only two 2-cells. After the one column is added to the other this row contains only one 1 and this 1 is common to the row and column removed. Hence the first type of step has the effect of reducing the rank of H_2 by 1.

It also has the effect of removing the column of H_1 corresponding to the 1-cell suppressed. This 1-cell is on the boundary of a 2-cell. Hence the 0-circuit represented by the column removed is linearly dependent on the columns corresponding to the other 1-cells of the boundary of this 2-cell. Hence the removal of this column leaves the rank of H_1 unaltered.

The first type of step thus changes ϱ_2 and ϱ_1 into $\varrho_2 - 1$ and ϱ_1 respectively. It obviously changes α_0, α_1, and α_2 into α_0, $\alpha_1 - 1$ and $\alpha_2 - 1$ respectively. A similar argument shows that the second type of step changes ϱ_2 and ϱ_1 into ϱ_2 and $\varrho_1 - 1$ respectively and also changes α_0, α_1, α_2 into $\alpha_0 - 1$, $\alpha_1 - 1$, and α_2 respectively. Hence the numbers

$$\alpha_0 - \alpha_1 + \alpha_2$$
$$\alpha_1 - \varrho_1 - \varrho_2$$
$$\alpha_2 - \varrho_2$$

are the same for $\bar{C}_2$ as for C_2. This is a special case of the more general theorem, to be proved later, that these numbers are invariants of C_2 under the group of all homeomorphisms.

Manifolds and 2-Circuits

17. By the *boundary* of a 2-dimensional complex C_2 is meant the one-dimensional complex containing each 1-cell of C_2 which is incident with an odd number of 2-cells of C_2.

By a *2-dimensional circuit* or a *2-circuit* is meant a 2-dimensional complex C_2 without a boundary such that any 2-dimensional complex whose 2-cells are a subset of the 2-cells of C_2 has a boundary. Thus any 2-dimensional complex in which each 1-cell is incident with an even number of 2-cells is evidently a 2-circuit or a set of 2-circuits having only 0-cells and 1-cells in common.

A 2-dimensional complex containing no 2-circuits is called a 2-dimensional *tree.*

18. By a *neighborhood* of a point P of a complex C_2 is meant any set S of 0-cells, 1-cells and 2-cells composed of points of C_2 and such that any set of points of C_2 having P as a limit point contains points on the cells of S. Thus any triangle star of a regular complex is a neighborhood of its center. Since (cf. § 14) any point of a complex C_2 can be made a vertex of a regular subdivision of C_2, the process of regular subdivision gives an explicit method of finding a neighborhood of any point of C_2.

19. If C_2 is a 2-circuit of which every point has a neighborhood which is a 2-cell, then the set of all points on C_2 is called a *closed two-dimensional manifold.** If $\overline{C}_2$ is a regular subdivision of a 2-circuit C_2 then it is evident that C_2 defines a manifold if and only if it is true that for each vertex P of $\overline{C}_2$ the incidence relations between the 1-cells and 2-cells of $\overline{C}_2$ which are incident with P are the same as those between the 0-cells and 1-cells of a 1-circuit.

A set of points obtainable from a closed two-dimensional manifold by removing a finite number of 2-cells no two of which have an interior or boundary point in common is called an *open two-dimensional manifold.* In the rest of this chapter the term manifold will mean "closed manifold" unless the opposite is specified.

20. The simplest example of a two-dimensional manifold

* We use this term rather than "surface" in order to have a terminology which may be used without confusion in Algebraic Geometry. In the latter science the real and complex points of a surface constitute a four-dimensional manifold.

is one determind by a complex consisting of two 0-cells, two 1-cells and two 2-cells, each 0-cell being incident with both 1-cells and each 1-cell with both 2-cells. Thus the matrices defining the manifold are

$$H_0 = \| 1 \quad 1 \|, \qquad H_1 = H_2 = \left\| \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right\|.$$

Such a manifold is called a *two-dimensional sphere.* It is easily seen to be homeomorphic with the surface of a tetrahedron.

21. A simple example of an open manifold, M_2, is obtained from a rectangle $ABCD$ (Fig. 4) by setting up a 1–1 continuous correspondence F between the 1-cells AB and CD and their ends in such a way that A corresponds to D and B corresponds to C, and then regarding the pairs of points

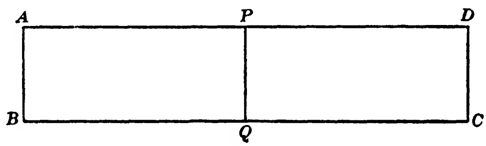

Fig. 4.

which correspond under F each as a single point of M_2. This open manifold is called a *tube* or a *cylindrical surface.* That it satisfies the definition of an open manifold is easily proved by dividing the rectangle into 2-cells by a 1-cell joining a point P of the side AD to a point Q of the side BC. It is bounded by the two curves formed from the 1-cells AD and BC respectively.

Let a (1–1) continuous correspondence F^1 be set up between the 1-cells AD and BC and their ends in such a way that A corresponds to B, P to Q, and D to C. A closed manifold T is defined by regarding as single points of T each pair of points which correspond either under F or under F^1. The four points A, B, C, D thus coalesce to one point of T. This manifold is called an *anchor ring* or *torus.*

22. If a correspondence G between the 1-cells AB and CD and their ends is set up in such a way that A corresponds to C

and B to D, an open manifold M is obtained by regarding each pair of points which correspond under G as a single point of M. This open manifold is called the *Möbius band.** A model is most simply constructed by taking a rectangle, giving it a half-twist and bringing opposite edges together. Thus the rectangle in Fig. 4 represents a Möbius band (Fig. 5) if we regard as identical

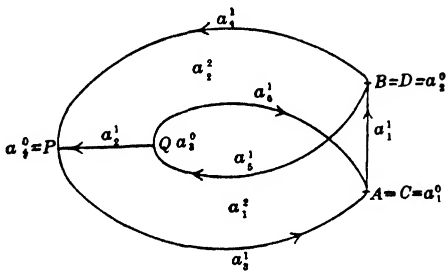

Fig. 5.

the two vertices labelled a_1^0, the two edges labelled a_1^1 and the two vertices a_2^0. If the rectangle be divided into two 2-cells by the 1-cell a_2^1 joining the two points a_3^0 and a_4^0 we obtain the following matrices which describe the Möbius band.

$$H_0 = \quad 1 \quad 1 \quad 1 \quad 1 \quad ,$$

$$H_1 = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{Vmatrix}, \quad H_2 = \begin{Vmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

23. The Möbius band is bounded by the 1-circuit (0, 0, 1, 1, 1, 1). If a 2-cell be introduced which is bounded by this 1-circuit a complex is obtained whose matrices H_0 and H_1 are the same as H_0 and H_1 for the Möbius band, while

* Cf. A. F. Möbius, Gesammelte Werke, Vol. 2, pages 484 and 519.

$$H_2 = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}.$$

The set of points on this complex is a manifold homeomorphic with the projective plane. Another set of matrices for the projective plane and some discussion of its Analysis Situs properties will be found in Veblen and Young's Projective Geometry, Vol. II, Chap. IX.

24. The operation of adding two one-dimensional complexes. modulo 2, which was defined in § 15, Chap. I may be extended to two dimensions as follows. Let C_2 und C_2^1 be two 2-dimensional complexes each of which is a sub-complex of a given complex C_2^2. By

$$C_2 + C_2^1 \text{ (mod. 2)}$$

is meant the complex composed of those 2-cells and their boundaries which are in either of C_2 and C_2^1 but not in both. This operation has the obvious property that if C_2 and C_2^1 are 2-circuits $C_2 + C_2^1$ (mod. 2) is also a 2-circuit or set of 2-circuits.

25. Let a sphere, S, be decomposed into cells by the process described in § 11 and let $s_1^2, s_2^2, \cdots, s_p^2$ be p of the 2-cells so obtained. Let $T^1, T^2, \cdots, T^p$ be p anchor rings no two of which have a point in common and which are such that s_i^2 $(i = 1, 2, \cdots, p)$ is a 2-cell of T^i while T^i and S have no other points in common than those of s_i^2 and its boundary. The set of all points on the 2-circuit,

$$(1) \qquad M_2 = S + T^1 + T^2 + \cdots + T^p \quad \text{(mod. 2)},$$

is called a *sphere with p handles,* or *an orientable manifold of genus p,* or *an orientable manifold of connectivity* $2p+1$. The proof that the set of points on M_2 is a manifold is made by subdividing it into 2-cells. By the same device it

is easy to prove that a sphere with one handle is an anchor ring.

26. If one of the anchor rings T^i in the last section is replaced by a projective plane, the 2-circuit M_2 is easily seen to define a manifold. We shall refer to this as a *one-sided manifold of the first kind of genus* $p-1$, or *of connectivity* $2p$. It is easy to verify that a projective plane is a one-sided manifold of the first kind of genus zero.

If two of the manifolds T^i are projective planes and the rest are anchor rings the 2-circuit M_2 again defines a manifold. This is called a *one-sided manifold of the second kind of genus* $p-2$, or *of connectivity* $2p-1$.

In this section and the last one the terms connectivity and genus are used in such a way that

$$R_1 - 1 = 2p + k$$

where R_1 is the connectivity, p is the genus, and $k = 0$ for an orientable manifold, k 1 for an one-sided manifold of the first kind, and k 2 for an one-sided manifold of the second kind.

27. The fundamental problem of two-dimensional Analysis Situs is that of classifying all two-dimensional manifolds. The solution of this problem is found by proving: (1) that for every manifold there is an integer R_1, the connectivity (cf. § 29), which is an invariant under the group of all homeomorphisms; (2) that there is an invariant property, that of "orientableness"; and (3) that any two manifolds which have the same connectivity and are both orientable or both non-orientable are homeomorphic. From this it will follow that the examples given in §§ 25 and 26 include all two-dimensional manifolds.

The proof of the propositions (1) and (2) will be given in considerable detail in the following pages because it is the basis of important generalizations to n-dimensions. The third proposition is covered more summarily because methods of proving it are well known and there is no possibility of generalizing it directly to n-dimensions. There is no known

system of invariants or invariant properties of n-dimensional manifolds which will characterize a manifold completely even in the three-dimensional case.

The Connectivity R_1

28. The boundary of any of the 2-cells a_i^2 which enter into the definition of a complex C_2 is given by one of the columns of the matrix $\boldsymbol{H}_2$. The boundary of the complex determined by two of these 2-cells is evidently the sum (mod. 2) of the boundaries of the 2-cells, and therefore is a 1-circuit or set of 1-circuits composed of cells a_i^0 and a_j^1 of C_2. By a repetition of these considerations it follows that the boundary of any two-dimensional complex composed of cells of C_2 is a 1-circuit or set of 1-circuits which is the sum (mod. 2) of the boundaries of the 2-cells of the complex. Hence a symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ for such a boundary is linearly dependent (mod. 2) on the columns of $\boldsymbol{H}_2$.

Moreover if any symbol $(x_1, x_2, \cdots, x_{\alpha_1})$ is linearly expressible in terms of the columns of $\boldsymbol{H}_2$ this expression determines a set of 2-cells of C_2 such that the symbol for the sum of their boundaries is $(x_1, x_2, \cdots, x_{\alpha_1})$. Hence *a necessary and sufficient condition that a set of 1-circuits composed of cells of C_2 shall bound a complex composed of cells of C_2 is that its symbol shall be linearly dependent on the columns of $\boldsymbol{H}_2$.*

29. By § 25, Chap. I the number of solutions of the equations $(\boldsymbol{H}_1)$ in a complete set is $\alpha_1 - \varrho_1$. So this is the number of 1-circuits in a complete set. If ϱ_2 is the rank of $\boldsymbol{H}_2$, the 1-circuits which bound complexes composed of cells of C_2 are all linearly dependent on ϱ_2 such 1-circuits. Hence a complete set of solutions of $(\boldsymbol{H}_1)$ is obtained by adjoining the symbols for $\alpha_1 - \varrho_1 - \varrho_2$ 1-circuits or sets of 1-circuits to ϱ_2 linearly independent columns of $\boldsymbol{H}_2$. Let us set

$$(1) \qquad R_1 - 1 = \alpha_1 - \varrho_1 - \varrho_2.$$

Hence there exist $R_1 - 1$ 1-circuits or sets of 1-circuits $C_1^1, C_1^2, \cdots, C_1^{R_1-1}$ such that every 1-circuit composed of

1-cells of C_2 is linearly dependent (mod. 2) on these and on the boundaries of 2-cells of C_2.

It can be so arranged that each of $C_1^1, C_1^2, \cdots, C_1^{R_1-1}$ is a single 1-circuit. For if C_1^1 represents more than one 1-circuit it is the sum (mod. 2) of these 1-circuits and at least one of these must be linearly independent of $C_1^2, \cdots, C_1^{R_1-1}$ and the bounding circuits, for otherwise C_1^1 would itself be linearly dependent on them. Let C_1^1 be replaced by this non-bounding 1-circuit. In like manner, there is at least one one among the 1-circuits represented by C_1^2 which is linearly independent of $C_1^1, C_1^3, \cdots, C_1^{R_1-1}$ and the bounding 1-circuits, for otherwise C_1^2 would be linearly dependent on them. Let C_1^2 be replaced by this 1-circuit and let a similar treatment be applied to C_1^3, and so on. A set of 1-circuits thus determined is called a *complete set of non-bounding 1-circuits*. It has the properties: (1) There is no two-dimensional complex composed of cells of C_2 which is bounded by these 1-circuits or any subset of them. (2) If C_1 is any 1-circuit composed of cells of C_2 there is a two-dimensional complex composed of cells of C_2 which is bounded either by C_1 alone or by C_1 and some of the circuits C_1^i $(i = 1, 2, \cdots, R_1 - 1)$. The number, R_1, is called the *connectivity* of the complex C_2, or, when it is necessary to distinguish it from the other connectivities R_i which are defined later, the *linear connectivity*.

30. Now suppose that C_2 consists of a single 2-circuit. In this case the sum (modulo 2) of the 1-circuits bounding the 2-cells is $(0, 0, \cdots, 0)$. This constitutes one linear relation among the columns of H_2. There cannot be more than one such relation, for this would imply that a subset of the 2-cells satisfied the definition of a 2-circuit. Hence the rank of H_2 is $\alpha_2 - 1$. Thus we have

(2) $$\rho_2 = \alpha_2 - 1,$$

and from § 20, Chap. I we have

(3) $$\rho_1 = \alpha_0 - R_0.$$

But since any 2-circuit is connected, $R_0 = 1$. Hence on combining (2) and (3) with (1) of § 29 we obtain

$$(4) \qquad \alpha_0 - \alpha_1 + \alpha_2 = 3 - R_1 .$$

This is one of the generalizations of Euler's well-known formula for a polyhedron.

31. Since a two-dimensional closed manifold is the set of points on a particular kind of 2-circuit the formula (4) of § 30, gives the relation between the connectivity R_1 and the characteristic of any two-dimensional complex defining a closed manifold. In the case of an open manifold, M_2, according to § 19, the boundary consists of a number of curves. Call this number B_1. Of these curves, $B_1 - 1$ are linearly independent because otherwise they would be the boundary of a manifold contained in M_2, contrary to definition. As in § 29, a complete set of 1-circuits in the complex C_2 defining M_2 may be taken to consist of ρ_2 bounding 1-circuits and $R_1 - 1$ non-bounding 1-circuits; and of the latter, $B_1 - 1$ may be taken to be circuits of the boundary of M_2. Hence if $R_1 - B_1 = \bar{R}_1 - 1$, the non-bounding circuits in the complete set comprise $B_1 - 1$ from the boundary and $\bar{R}_1 - 1$ others.

If C_2 be modified by introducing B_1 2-cells each bounded by one of the B_1 1-circuits of the boundary, C_2 becomes a 2-circuit C_2^1 of $\alpha_2 + B_1$ 2-cells, α_1 1-cells, and α_0 0-cells in which $B_1 - 1$ of the non-bounding circuits of C_2 have become bounding circuits. Hence C_2^1 has the connectivity $\bar{R}_1$. Hence

$$\alpha_0 - \alpha_1 + \alpha_2 + B_1 = 3 - \bar{R}_1 ,$$

and

$$\begin{aligned} \alpha_0 - \alpha_1 + \alpha_2 &= 3 - \bar{R}_1 - B_1 \\ &= 2 - R_1 \end{aligned}$$

which is the formula for the characteristic of a complex defining an open manifold of two dimensions. The same formula holds for any connected two-dimensional tree, as follows from (1) and (3) and the fact that $\rho_2 = \alpha_2$.

Singular Complexes

32. The cells a_i^0, a_j^1, a_k^2 which enter into the definition of a complex are all non-singular and their boundaries are also non-singular. This restriction was necessary in order to obtain the theorem of § 6 that the matrices H_0, H_1, H_2 fully determine the complex. In many applications, however, it is desirable to drop the restriction that the boundaries of the cells referred to in the matrices H_i shall be non-singular. The results of the theory of matrices can in general be applied whenever it is possible to subdivide the cells having singular boundaries by means of a finite number of 0-cells and 1-cells in such a way as to obtain a complex of non-singular cells with non-singular boundaries.

For example, in § 21 the anchor ring was defined as consisting of one 0-cell, represented by the four vertices of the rectangle, two 1-cells represented by its pairs of opposite edges, and one 2-cell. The matrices of incidence relations of these cells are

$$H_0 = \| 1 \| . \qquad H_1 = \| 0 \ 0 \| . \qquad H_2 = \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\| .$$

Thus $\varrho_0 = 1$, $\varrho_1 = 0$, $\varrho_2 = 0$, $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$. Hence

$$\begin{aligned} R_1 &= 3 - (\alpha_0 - \alpha_1 + \alpha_2) = 3 \\ &= \alpha_1 - \varrho_1 - \varrho_2 + 1 . \end{aligned}$$

If the rectangle is subdivided into triangles so that a non-singular complex is obtained it will be found that the same value for R_1 will be obtained from the non-singular complex as from the singular one.

33. The notion of a singular complex on a one-dimensional complex, as defined in § 8, Chap. I, can be generalized directly to two dimensions as follows:

Let C_2 be a two-dimensional complex, C' a generalized complex of zero, one or two dimensions*, and F a correspondence

* The definition may be extended so that C' is of any number of dimensions.

in which each point of C' corresponds to one point of a set of points $[P]$ of C_2 while each P is the correspondent of one or more points of C'. If C' is of one or two dimensions we require F to be continuous. Under these conditions, any point X of C' associated with the P to which it corresponds under F is called *a point on* C_2; it is referred as the *image* of X under F and is uniquely denoted by $F(X)$; it is said to *coincide* with P and P is said to *coincide* with it. The point $F(X_1)$ is called a limit point of the points $F(X)$ if X_1 is a limit point of the points X. The set of all points $F(X)$ on C_2 is in a $(1-1)$ continuous correspondence with the points of C' and thus constitutes a complex C'' identical in structure with C'. The complex C'' is said to be *on* C_2. If any of the points P is the correspondent under F of more than one point of C', C'' is called a *singular complex on* C_2 and the point P in question is called a *singular point.* If F is $(1-1)$, C'' is said to be *non-singular.* A cell of C'' is said to *coincide with* a cell of C_2 if and only if the two cells are in $(1-1)$ continuous correspondence, under F.

In case C'' is two-dimensional and such that there is at least one point of C'' on each point of C_2 and if, furthermore, there exists for every point of C'' a neighborhood which is a non-singular complex on C_2, then C'' is said to *cover* C_2. In case the number of points of C'' on each point of C_2 is finite and equal to n, C'' is said *to cover* C_2 n *times* (cf. § 9, Chap. I).

34. Any 2-circuit which is not a manifold can be regarded as a singular manifold. For let C_2 be an arbitrary 2-circuit. Each of its edges, a_i^1, is incident with an even number, $2n_i$ of 2-cells. These 2-cells may be grouped arbitrarily in n_i pairs no two of which have a 2-cell in common; let these be called the *pairs of 2-cells associated with* a_i^1. Let C_2' be a 2-circuit on C_2 such that (1) there is one and but one 2-cell of C_2' coinciding with each 2-cell of C_2, (2) there are n_i 1-cells of C_2' coinciding with each 1-cell a_i^1 of C_2, each of the n_i 1-cells being incident with a pair of 2-cells of C_2' which coincide with one of the pairs of 2-cells associated

with a_i^1, and (3) there is one 0-cell of C_2' coincident with each 0-cell a_i^0 of C_2, this 0-cell being incident with all the 1-cells of C_2' which coincide with 1-cells of C_2 incident with a_i^0. Thus C_2' has two 2-cells incident with each of its 1-cells.

The incidence relations of the 1-cells and 2-cells of C_2' which are incident with a vertex a_i^0 of C_2' are the same as those of the 0-cells and 1-cells of a linear graph and since there are just two 2-cells incident with each 1-cell this linear graph consists of a number of 1-circuits having no points in common. Let any set of 1-cells and 2-cells of C_2' which are incident with a_i^0 and whose incidence relations with one another are those of a 1-circuit be called *a group associated with* a_i^0. Let C_2'' be a 2-circuit on C_2' such that (1) there is one and but one i-cell ($i = 1, 2$) of C_2'' coinciding with each i-cell of C_2', (2) the incidence relations between the 1-cells and 2-cells of C_2'' are the same as those between the cells of C_2' with which they coincide, and (3) there is one 0-cell of C_2'' for each group associated with each vertex a_i^0 of C_2' and this 0-cell is coincident with a_i^0 and incident with those and only those 1-cells and 2-cells of C_2'' which coincide with 1-cells and 2-cells of the group. The set of points on the complex C_2'' is a two-dimensional manifold, by § 19, and C_2'' is a singular complex on C_2. Hence C_2 may be obtained by coalescing a certain number of 1-cells and 0-cells of a manifold.

Bounding and Non-bounding 1-Circuits

35. Having defined what is meant by saying that a complex C_n ($n = 0, 1, 2$) is on a complex C_2, we can now state and solve the problem of bounding and non-bounding circuits in a perfectly general form: *Given any set of 1-circuits K_1 on a complex C_2, does there exist a two-dimensional complex K_2 on C_2 which is bounded by K_1?*

In spite of the generality of the complex K_1, and because of the generality of K_2, this problem is free from many of the difficulties inherent in such point-set theorems as those

of Schoenflies and Jordan. This will be illustrated by the simple case considered in the next section.

36. Any closed curve, singular or not. which is on a 2-cell a^2 and its boundary but does not pass through every point of a^2 is the boundary of a 2-cell on a^2. Let c be the given curve and O a point of a^2 not on c. Let OX be the straight 1-cell joining O to a variable point X of c. Let O' be a point interior to a triangle t of a Euclidean plane and let X' be a variable point of the boundary of this triangle. Let F be a continuous (1 1) correspondence between the set of points $[X']$ and the set of points $[X]$. If we let each point of $O'X'$ correspond to the point of OX which divides it in the same ratio, a continuous correspondence F' is defined in which each point of the interior and boundary of the triangle t corresponds to one point of a^2. By § 1 there is thus defined a 2-cell (in general, singular) which is bounded by c.

It is not essential that O shall not coincide with a point of c, for in case X coincides with O the interval OX may be taken to be a singular one coinciding with O. Hence we have without restrictions the theorem that any closed curve on a 2-cell a is the boundary of a 2-cell on a.

The theorem may be generalized slightly as follows· *Any curve c on a triangle star and its boundary* (§ 14) *is the boundary of a 2-cell on the triangle star*. The 2-cell is constructed as above. taking the center of the triangle star as O.

Congruences and Homologies, Modulo 2

37. Before going on to the solution of the problem stated in § 35, let us introduce a notation which is adapted from that of Poincaré. We shall say that a complex C_n ($n = 1, 2$) is *congruent* (*mod. 2*) to a set of $(n-1)$-circuits C_{n-1} if and only if C_{n-1} is the boundary of C_n. This is represented by the notation

(1) $$C_n \equiv C_{n-1} \quad (\text{mod. } 2).$$

In case C_{n-1} fails to exist, so that C_n is a set of n-circuits, C_n is said to be *congruent to zero* (*mod. 2*) and (1) is replaced by

(2) $$C_n \equiv 0 \pmod{2}.$$

Expressions of the form (1) and (2) are called *congruences (mod. 2)*. They have been defined thus far only for $n = 1$ and $n = 2$, but these definitions will apply for all values of n as soon as the terms complex, n-circuit, and boundary of an n-dimensional complex have been defined for all values of n.

Both in the one- and two-dimensional cases it is evident that when two complexes are added (mod. 2) the boundary of the sum is the sum (mod. 2) of the boundaries. Hence the sum (mod. 2) of the left-hand members of two congruences is congruent to the sum (mod. 2) of the right-hand members. Or, more generally, *any linear combination (mod. 2) of a number of valid congruences (mod. 2) of the same dimensionality is a valid congruence (mod. 2).*

38. With respect to a complex C a complex C_{n-1} is said to be *homologous* to zero (mod. 2) if and only if it is the right-hand member of a congruence such as (1) in which C_n represents a complex on C. This relation is indicated by

(3) $$C_{n-1} \sim 0 \pmod{2}.$$

Thus

$$C_0 \sim 0 \pmod{2}$$

means that C_0 represents a set of 0-circuits which bound a one-dimensional complex on C, and

$$C_1 \sim 0 \pmod{2}$$

means that C_1 represents a set of 1-circuits on C which bound a two-dimensional complex on C. Thus in every case, (3) implies

(4) $$C_{n-1} \equiv 0 \pmod{2},$$

but (4) does not imply (3).

From the corresponding proposition in the last section it follows at once that *any linear combination (mod. 2) of a set of valid homologies (mod. 2) is a valid homology (mod. 2).* A homology,

(5) $$C_{n-1} + C'_{n-1} \sim 0 \pmod{2},$$

is also written

(6) $$C_{n-1} \sim C'_{n-1} \pmod{2}.$$

The homology (6) evidently means that there exists a complex C_n on C which is bounded by C_{n-1} and C'_{n-1}.

If $\bar{C}_1$ is a 1-circuit obtained by introducing new vertices in a 1-circuit C_1, it is evident that

$$C_1 \sim \bar{C}_1 \quad (\text{mod. } 2),$$

because C_1 and $\bar{C}_1$ bound a singular two-dimensional complex coincident with them both.

The Correspondence A

39. The first step toward the solution of the problem of § 35 will be to show that if $\bar{C}_2$ is a regular subdivision of C_2, then for any 1-circuit K_1 on C_2 there is a set of 1-circuits K'_1 *composed of cells of* $\bar{C}_2$ such that

$$(1) \qquad K_1 \sim K'_1 \quad (\text{mod. } 2).$$

This has the consequence that any homology among 1-circuits can be replaced by one in which each 1-circuit is composed of cells of $\bar{C}_2$; and the problem of § 35 is reduced to that of finding a necessary and sufficient condition that $K'_1 \sim 0$ (mod. 2) if K'_1 represents a set of 1-circuits composed of cells of $\bar{C}_2$. The next three sections aim at establishing the homology (1).

40. Let K be a one- or two-dimensional complex on a two-dimensional complex C_2. Let $\bar{C}_2$ be a regular subdivision of C_2. Let a definition of distance and straightness be introduced relative to $\bar{C}_2$, and let $\tilde{C}_2$ be a regular subdivision of $\bar{C}_2$ whose 1-cells are all straight. The triangle stars of $\tilde{C}_2$ constitute a set of overlapping neighboorhoods such that every point of $\tilde{C}_2$ is interior to at least one of these neighborhoods. Hence by simple continuity considerations (Heine-Borel theorem) K can be subdivided, by introducing new vertices if it is of one dimension, or by the process of regular subdivision (§ 13) if it is of two dimensions into a complex $\bar{K}$ such that for each 1-cell or 2-cell of $\bar{K}$ there is a triangle star of $\tilde{C}_2$ to which it is interior.

Those of the triangle stars of $\bar{C}_2$ whose centers are vertices of C_2 have the property that any point of C_2 is either interior to one such triangle star or on the boundaries of 2-cells from two or more such triangle stars. Let us designate as *a correspondence A* any correspondence of the vertices of $\bar{K}$ with those of C_2 by which each vertex of $\bar{K}$ which is interior to a triangle star of $\bar{C}_2$ having a vertex of C_2 as center corresponds to this center, and each vertex of $\bar{K}$ which is on the boundary of two or more such triangle stars corresponds to the center of one of them.* Thus a correspondence A determines a unique vertex of C_2 for each vertex of $\bar{K}$.

This construction is such that any triangle star of $\bar{C}_2$ which contains a vertex of K has the 0-cell of C_2 to which this vertex corresponds on its interior or boundary. Moreover any two vertices of K which are ends of the same 1-cell of $\bar{K}$ coincide with points of the same triangle star of $\bar{C}_2$ and hence correspond to points of C_2 of the interior or boundary of this triangle star. Hence they correspond either to the same vertex of $\bar{C}_2$ or to the two ends of a 1-cell of C_2 (Cf. § 14). In case K is two-dimensional it follows similarly that any three vertices of K incident with the same 2-cell of K correspond to one or more vertices of a single 2-cell of $\bar{C}_2$.

41. Let the 0-cells, 1-cells and 2-cells of C_2 be denoted by $c_1^0, c_2^0, \cdots, c_{\alpha_0}^0$; $c_1^1, c_2^1, \cdots, c_{\alpha_1}^1$; and $c_1^2, c_2^2, \cdots, c_{\alpha_2}^2$ respectively; and those of $\bar{K}$ by $k_1^0, k_2^0, \cdots, k_{\beta_0}^0$; $k_1^1, k_2^1, \cdots, k_{\beta_1}^1$; $k_1^2, k_2^2, \cdots, k_{\beta_2}^2$ respectively. Having fixed on a correspondence A between the vertices of K and those of $\bar{C}_2$, let each 0-cell k_i^0 be joined by a straight 1-cell b_i^1 to the corresponding vertex of $\bar{C}_2$ in case k_i^0 does not coincide with its correspondent; and if k_i^0 does coincide with its correspondent let it be joined to its correspondent by a singular 1-cell b_i^1 coinciding with it.

The two ends of a 1-cell k_i^1 are thus joined by two 1-cells b_i^1 and b_k^1 either to the same vertex of $\bar{C}_2$ or to the two ends

* This is essentially the same as requiring (with Alexander, in the paper cited in our preface) that each vertex of K shall correspond to the nearest vertex of $\bar{C}_2$, or to one of the nearest if there are more than one.

of a 1-cell c_p^1 of C_2. In the first case k_i^1, b_j^1 and b_k^1 are the 1-cells of a 1-circuit and in the second case k_i^1, b_j^1, b_k^1 and c_p^1 are the 1-cells of a 1-circuit. In either case there is a single triangle star of $\bar{C}_2$ which, with its boundary, contains the 1-circuit. Therefore by § 36 the 1-circuit bounds a 2-cell b_i^2 on C_2. *Thus each 1-cell k_i^1 of K determines a 2-cell b_i^2.* The complex composed of the 2-cells b_i^2 and their boundaries is called B_2.

42. The incidence relations between the 1-cells b_j^1 and the 2-cells b_i^2 of B_2 are the same as the incidence relations between the 0-cells and 1-cells of K. Hence, in particular, if K is a 1-circuit or set of 1-circuits, K_1, the sum (mod. 2) of the boundaries of the 2-cells b_i^2 contains none of the 1-cells b_j^1. Hence the boundary of B_2 can consist only of cells of K_1 and of C_2. Hence the boundary of B_2 is either K_1 alone or K_1 and a set of 1-circuits composed of cells of C_2. Let the latter set of 1-circuits be denoted by K_1'.

Hence we have the congruence.

(2) $$B_2 \equiv K_1 + K_1' \quad (\text{mod. } 2)$$

in which K_1' is either zero or a set of 1-circuits composed of cells of C_2. From this there follows the homology

(1) $$K_1 \sim K_1' \quad (\text{mod. } 2)$$

which we have been seeking.

43. If K_1' is zero the question as to whether K_1 satisfies a homology

(3) $$K_1 \sim 0 \quad (\text{mod. } 2)$$

is answered in the affirmative. In any other case, since K_1' is composed of cells of C_2 it is represented by a symbol $(x_1, x_2, \cdots, x_{\alpha_1})$. If this symbol is linearly dependent on the columns of the matrix H_2 for C_2,

$$K_1' \sim 0 \quad (\text{mod. } 2)$$

according to § 28. Moreover K_1' cannot bound a complex composed of cells of C_2 unless its symbol $(x_1, x_2, \cdots. x_{\alpha_1})$ is

linearly dependent on the columns of H_2. If, therefore, we can prove that K_1' cannot bound any complex on C_2 unless it bounds one composed of cells of $\bar{C}_2$, it will follow that (3) is satisfied if and only if $(x_1, x_2, \cdots, x_{\alpha_1})$ is linearly dependent on the columns of H_2. This we proceed to do, thus completing the solution of the problem stated in § 35.

44. Let us return to the notations of §§ 40 and 41 and suppose that K is a two-dimensional complex K_2. The three 1-cells k_i^1, k_j^1, k_l^1 of K_2 incident with a 2-cell k_p^2 of $\bar{K}_2$ have been seen to determine three 2-cells b_i^2, b_j^2, b_l^2. These 2-cells are incident by pairs with the 1-cells joining the three vertices of k_p^2 to their correspondents under the correspondence A. The vertices of $\bar{C}_2$ to which the vertices of k_p^2 correspond are either the three vertices of a 2-cell c_q^2 of C_2 or the two ends of a 1-cell of C_2 or a single 0-cell of C_2. In the first case the 2-cells, $k_p^2, b_i^2, b_j^2, b_l^2$ and c_q^2 are the 2-cells of a sphere; in the second and third cases the 2-cells k_p^2, b_i^2, b_j^2, and b_l^2 are the 2-cells of a sphere. Let the sphere which is thus in every case determined by k_p^2 be denoted by S_2^p.

A 2-cell b_i^2 is in an odd number of these spheres if and only if it is incident with a 1-cell k_i^1 of the boundary of $\bar{K}_2$. Hence the result of adding the spheres S_2^p to K_2 (mod. 2) is either zero or a complex K_2' the 2-cells of which are either 2-cells of C_2 or 2-cells b_i^2 determined by the 1-cells of the boundary of $\bar{K}_2$. In particular, if $\bar{K}_2$ is a 2-circuit, either $\bar{K}_2$ is the sum (mod. 2) of the spheres S_2^p or K_2' is composed entirely of cells of C_2.

45. If K_2 has a boundary, so that

$$(4) \qquad K_2 \equiv K_1 \quad (\text{mod. } 2),$$

the result of the last section is that by adding a number of congruences,

$$(5) \qquad S_2^p \equiv 0 \quad (\text{mod. } 2),$$

to (4) we obtain a congruence,

$$(6) \qquad K_2' \equiv K_1 \quad (\text{mod. } 2),$$

such that all 2-cells of K_2' are either 2-cells of C_2 or 2-cells b_i^2 determined by the boundary K_1 of K_2'. The complex B_2'' composed of the latter 2-cells and their boundaries is such that

$$B_2'' \equiv K_1 + K_1'' \pmod{2} \tag{7}$$

where K_1'' is composed of 0-cells and 1-cells of C_2. On adding (6) and (7) we obtain a congruence

$$K_2' + B_2'' \equiv K_1'' \pmod{2} \tag{8}$$

in wich the left-hand member represents a complex composed only of cells of C_2.

46. It is now easy to obtain the result required at the end of § 43, namely that if a set of 1-circuits K_1' is composed of cells of $\bar{C}_2$, then

$$K_1' \sim 0 \pmod{2}$$

implies that K_1' is the boundary of a complex composed of cells of $\bar{C}_2$. Taking a complex bounded by K_1', we subdivide it as above, preliminary to setting up a correspondence A, and denote by K_1 the corresponding subdivision of K_1'. Thus we have a congruence like (4) of the last section, consequently one like (8) derived from it. But in this case the K_1'' constructed in the last section is easily seen to be identical with K_1'. Hence (8) states that K_1' is the boundary of a complex composed of cells of $\bar{C}_2$.

Invariance of R_1

47. An immmediate corollary of what has just been proved is that the 1-circuits $C_1^1, C_1^2, \cdots, C_1^{R_1-1}$ of a complete set (§ 29) of non-bounding 1-circuits of $\bar{C}_2$ are not connected by any homology of the form

$$C_1^{i_1} + C_1^{i_2} + \cdots + C_1^{i_k} \sim 0 \pmod{2} \tag{1}$$

in which the superscripts are distinct integers less than R_1. Moreover if K_1 is any 1-circuit on C_2 it satisfies a homology of the form

$$K_1 \sim C_1^{j_1} + C_1^{j_2} + \cdots + C_1^{j_p} \pmod{2} \tag{2}$$

in which the terms of the right-hand member represent 1-circuits of the complete set. For by § 42

$$(3) \qquad K_1 \sim K_1' \quad (\text{mod. } 2)$$

in which K_1' is zero or a set of 1-circuits composed of cells of $\bar{C}_2$, and by § 29 K_1' is homologous to a combination of 1-circuits of the complete set.

48. But if $K_1^1, K_1^2, \cdots, K_1^N$ is any set of 1-circuits such that (1) any 1-circuit is homologous to a linear combination of them and (2) there is no homology relating them, it is easily proved that $N = R_1 - 1$. For by the properties of the 1-circuits $C_1^1, C_1^2, \cdots, C_1^{R_1-1}$, there are N homologies like (2),

$$(4) \qquad K_1^j \sim C_1^{j_1} + C_1^{j_2} + \cdots + C_1^{j_p} \quad (\text{mod. } 2),$$

one for each value of j from 1 to N. If $N > R_1 - 1$ the right-hand members of (4) must satisfy a homology because there are only $R_1 - 1$ C_1^i's. But this is contrary to the property (2) of the K_1^i's. Hence $N > R_1 - 1$ is impossible. In like manner, inverting the roles of the K_1^i's and the C_1^i's, it follows that $R_1 - 1 > N$ is impossible. Hence $N = R_1 - 1$.

Any homeomorphism of C_2 obviously transforms a set of 1-circuits $K_1^1, K_1^2, \cdots, K_1^N$ satisfying the conditions (1) and (2) into a set of 1-circuits satisfying the same conditions. Since $N = R_1 - 1$ for every such set of 1-circuits, it follows that *R_1 is an Analysis Situs invariant of the complex C_2.*

49. It was proved in § 16 that the expression in the right-hand member of

$$R_1 - 1 = \alpha_1 - \varrho_1 - \varrho_2$$

is the same for C_2 as for $\bar{C}_2$. Now let C_2 be sudivided into any set of cells which form a non-singular complex K_2 on C_2, and let $\bar{K}_2$ be a regular subdivision of K_2. The complex $\bar{K}_2$ can replace $\bar{C}_2$ in the discussion above and hence $\bar{K}_2$ has the same connectivity, R_1, as $\bar{C}_2$. Hence K_2 and C_2 have the same connectivity. In other words any two complexes have the same connectivity if they are identical as sets of points and the cells of each are non-singular on the other.

It should perhaps be remarked that the relation between K_2 and C_2 may be quite complex in spite of the fact that each cell of K_2 is non-singular on C_2 and *vice versa*. For any 1-cell of K_2 may intersect any number of 1-cells of C_2 in an infinite set of points, and any 2-cell of K_2 may have an infinite set of regions in common with any 2-cell of C_2.

Invariance of the 2-Circuit

50. If K_2 and C_2 are related as described in the last section, *K_2 is a 2-circuit if and only if C_2 is a 2-circuit.* Since the relation between C_2 and K_2 is reciprocal this theorem will be established if we prove that if K_2 is a 2-circuit then C_2 is one. Also it is evident that C_2 or K_2 is a 2-circuit if and only if a regular subdivision of it is a 2-circuit. Hence we replace C_2 by its regular subdivision $\bar{C}_2$ as in § 40 and construct the spheres S_2^p as in § 44. By § 44 the result of adding the spheres S_2^p to $\bar{K}_2$ (mod. 2) is either zero or a set of 2-circuits composed of cells of $\bar{C}_2$. If it were zero the 2-circuit $\bar{K}_2$ would be the sum (mod. 2) of the spheres S_2^p. But this is impossible, as shown by the following theorem.

51. *There is no set of 2-circuits K_2^i on a 2-circuit C_2 such that* (1) *for each 2-circuit K_2^i there is a 2-cell of C_2 on which there is no point of K_2^i and* (2) *the sum* (*mod.* 2) *of the 2-circuits K_2^i is C_2.*

To prove this theorem, we suppose that there is a set of 2-circuits K_2^i having the property (1). We let these 2-circuits take the place of K in § 40, make the regular subdivision of C_2 into $\bar{C}_2$ and K_2^i into $\bar{K}_2^i$, construct a correspondence A and obtain a set of spheres S_2^p (which, of course, must not be confused with those in § 50). When the spheres having 2-cells in common with one of the 2-circuits $\bar{K}_2^i$ are added to this $\bar{K}_2^i$ the result is either zero or a non-singular set of 2-circuits composed of cells of $\bar{C}_2$. But since C_2 is a 2-circuit the only 2-circuit composed of its cells is $\bar{C}_2$ itself. Since there is one 2-cell of C_2 which contains no point of K_2^i it follows that the sum of $\bar{K}_2^i$ and the spheres S_2^p determined by its 2-cells is zero.

Obviously if each of two 2-circuits is such that the sum (mod. 2) of it and the spheres S_2^p determined by its 2-cells is zero the same is true of the sum (mod. 2) of the two 2-circuits. Hence the sum of all the 2-circuits $\overline{K}_2^i$ has this property. On the other hand the 2-circuit $\overline{C}_2$ is such that the sum of it and the spheres S_2^p determined by its 2-cells is $\overline{C}_2$ itself. Hence the 2-circuits K_2^i do not have the property (2).

52. Letting the 2-circuit $\overline{K}_2$ and the spheres S_2^p of § 50 take the place of the 2-circuit C_2 and the 2-circuits K_2^i of § 51 it follows from the theorem of § 51 that $\overline{K}_2$ is not the sum (mod. 2) of the spheres S_2^p. Hence the sum (mod. 2) of $\overline{K}_2$ and the spheres S_2^p is a set of 2-circuits composed of cells of $\overline{C}_2$. We shall prove that these 2-circuits constitute $\overline{C}_2$. If they did not, let them be denoted by C_2', let c_j^2 be one of the 2-cells of $\overline{C}_2$ which is not on C_2', and let $\overline{K}_2$ be regularly subdivided into a complex K_2' which has at least one 2-cell which is interior to c_j^2.

The complex C_2' is composed of non-sigular cells on K_2' and hence C_2' and K_2' can replace $\overline{K}_2$ and $\overline{C}_2$ respectively in the construction used in § 50 for the spheres S_2^p. Thus a set of spheres can be found which when added to a regular subdivision of C_2' give a set of 2-circuits C_2'' composed of cells of a regular subdivision of K_2'. It follows from § 51 that C_2'' is not vacuous. Since $\overline{K}_2$ and its regular subdivisions are 2-circuits, C_2'' must be identical with the regular subdivision of K_2'. This is not possible unless there is a point of C_2' on each 2-cell of K_2'. But this implies that there is a point of C_2' on c_j^2, contrary to the hypothesis that c_j^2 is not a cell of C_2'. Hence C_2' coincides with $\overline{C}_2$, as we wished to prove.

Now by reversing the whole process we can show that any one of the 2-circuits that compose C_2' will yield a subdivision of the 2-circuit K_2. Hence there can be only one, and the proof of the theorem of § 50 is complete.

53. It is an obvious corollary of this theorem that the property of a two-dimensional complex, of being a 2-circuit, is an Analysis Situs invariant. For if C_2 and G_2 are two

complexes which are homeomorphic, the homeomorphism defines a non-singular complex K_2 on C_2 such that each cell of K_2 is the image of a cell of G_2. By definition, K_2 is a 2-circuit if and only if G_2 is a 2-circuit, and by the theorem of § 50 K_2 is a 2-circuit if and only if C_2 is a 2-circuit.

It is an obvious corollary of this result that the property of a complex, that it defines a manifold, is also an Analysis Situs invariant. In other words, any complex into which a manifold can be subdivided, satisfies the conditions laid down in § 19.

Matrices of Orientation

54. Let us now convert the 1-dimensional complex composed of the 0-cells and 1-cells of C_2 into an oriented one-dimensional complex in the fashion described in §§ 33 to 40 of Chap. I. The oriented 0-cells are

$$\sigma_1^0,\ \sigma_2^0,\ \cdots,\ \sigma_{\alpha_0}^0,$$

the 1-cells are

$$\sigma_1^1,\ \sigma_2^1,\ \cdots,\ \sigma_{\alpha_1}^1,$$

and the relations between them are given by the matrices E_0, E_1 satisfying the relation

$$E_0 \cdot E_1 = 0.$$

Each of the columns of H_2 is the symbol for a 1-circuit which, according to § 35, Chap. 1, determines two oriented 1-circuits. The symbol for either of these oriented 1-circuits may be obtained from the corresponding column of H_2 by changing some of the 1's to -1's. Hence by changing some of the 1's in H_2 to -1's there is determined a matrix

$$E_2 = \|\varepsilon_{ij}^2\| \qquad (i = 1, 2, \cdots, \alpha_1;\ j = 1, 2, \cdots, \alpha_2)$$

each column of which represents an oriented 1-circuit and is therefore a solution of the equations (E_1), § 42, Chap. 1. Hence

$$E_1 \cdot E_2 = 0.$$

As an example, a matrix E_2 for the tetrahedron in Fig. 1, page 2, is (cf. H_2 in § 4)

$$E_2 = \begin{Vmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{Vmatrix}.$$

A further example is furnished by the projective plane. for which (cf. §§ 22, 23)

$$E_1 = \begin{Vmatrix} -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}.$$

Note that the rank of E_2 for the tetrahedron is 3, or $\alpha_2 - 1$, and for the projective plane is 3, or α_2.

55. Let us denote the ranks of E_0, E_1, E_2 by r_0, r_1, r_2 respectively. We have seen that

$$r_0 = R_0 = \varrho_0,$$
$$r_1 = \varrho_1$$

and that in case C_2 is a 2-circuit,

$$\varrho_2 = \alpha_2 - 1.$$

It is impossible that r_2 should be less than $\alpha_2 - 1$ because this would imply a linear relation involving at most $\alpha_2 - 1$ columns, with relatively prime coefficients, and hence on reducing modulo 2, that the same statement was true of the columns of H_2, contrary to § 30. Hence there remain two possibilities

$$r_2 = \alpha_2 - 1$$

and

$$r_2 = \alpha_2$$

for any C_2 which is a 2-circuit. The examples in the last section show that both possibilities can be realized.

56. A 2-circuit C_2 such that $r_2 = \alpha_2 - 1$ has the property that if the boundaries of its 2-cells are converted into oriented 1-circuits in any way, they will satisfy a linear relation with integral coefficients. For the columns of E_2 represent a set of oriented 1-circuits, one bounding each 2-cell, and since $r_2 = \alpha_2 - 1$ they are subject to one linear relation,

$$(1) \qquad b_1 c_1 + b_2 c_2 + \cdots + b_{\alpha_2} c_{\alpha_2} = 0$$

in which the c's represent the columns of E_2 and the b's are positive or negative integers or zero. If the coefficients are divided by their highest common factor, and then reduced modulo 2, this relation must state that the sum of the columns of H_2 is zero. Hence the relation must involve all columns of E_2.

In case C_2 has the property that each 1-cell is incident with two and only two 2-cells (for example, if it is a manifold), if an oriented 1-cell σ_i^1 is to cancel out, the two oriented 1-circuits formed from the boundaries of the 2-cells incident with a_i^1 must appear in (1) with numerically equal coefficients. It follows that the coefficients of (1) are numerically equal and therefore that by removing a common factor (1) can be reduced to a form in which $b_i = \pm 1$.

Hence by multiplying some of the columns by -1, E_2 can be reduced to a form in which the sum of the columns is zero. The columns of E_2 then represent a set of oriented 1-circuits such that if σ^1 is any oriented 1-cell formed from a 1-cell of C_2, one of these 1-circuits contains σ^1 and another one contains $-\sigma^1$. Consequently if C_2 has the property that each of its 1-cells is incident with two and only two 2-cells, the boundaries of its 2-cells can be converted into oriented 1-circuits in such a way that their sum is zero.

Orientable Circuits

57. The theorem of the last section is that if $r_2 = \alpha_2 - 1$ for a 2-circuit C_2, the boundaries of the 2-cells of C_2 can be converted into oriented 1-circuits in such a way that they

satisfy a linear relation. If $r_2 = \alpha_2$ the boundaries of the 2-cells evidently cannot be thus oriented. In the first case C_2 is said to be *two-sided* or *orientable* and in the second case to be *one-sided* or *non-orientable.* A manifold is said to be *orientable* or *non-orientable* according as the complex defining it is or is not orientable. This extension of the term is justified by the theorems of §§ 58–60 below, according to which the complexes defining a given manifold M_2 are all orientable or all non-orientable.

This definition is equivalent to the one given in 1865 by A. F. Möbius, Über die Bestimmung des Inhaltes eines Polyëders, Werke, Vol. 2, p. 475; see also p. 519. The term "orientable" was suggested by J. W. Alexander as preferable to "two-sided" because the latter term connotes the separation of a three-dimensional manifold into two parts, the two "sides," by the two-dimensional manifold, whereas the property which we are dealing with is an internal property of the two-dimensional manifold.*

The intuitional significance of orientableness is perhaps best grasped by experiments with the well-known Möbius paper strip described in the article referred to above. These experiments can also be used to verify the theorems on deformation and on the indicatrix in Chap. V.

58. Suppose that a 2-cell a_i^2 of a complex C_2, the cells of which have been oriented in the manner described above, is separated into two 2-cells by a 1-cell a^1. The two new 2-cells are bounded by two 1-circuits which have a^1 in common. It is easily seen that if σ^1 is either of the oriented 1-cells formed from a^1, two oriented 1-circuits can be formed from the

* On the relation between orientableness and two-sidedness, see E. Steinitz, Sitzungsberichte der Berliner Math. Ges., Vol. 7 (1908), p. 35; and D. König, Archiv der Math. u. Phys., 3d Ser., Vol. 19 (1912), p. 214. The term orientable (orientierbar) has also been used by H. Tietze in an article in the Jahresbericht der Deutschen Math. Ver., Vol. 29 (1920), p. 95, which came to my attention while these lectures were in proof-sheets. This article contains a general discussion of orientability covering a number of the questions referred to in the beginning of Chap. V below, and also a useful collection of references.

boundaries of the two new 2-cells in such a way that one of them contains σ^1 and the other contains $-\sigma^1$. Hence the sum of these oriented 1-circuits is one of the two oriented 1-circuits which can be formed from the boundary of a_i^2.

The complex C_2 is converted into a new complex C_2' by intoducing the new 1-cell a^1 and subdividing a_i^2. The matrix E_2 of C_2' has one row and one column more than the matrix E_2 of C_2, and by the paragraph above can be converted into the matrix E_2 for C_2 by adding the two columns corresponding to the two new 2-cells and striking out the row corresponding to a^1. These operations evidently reduce the rank by 1. Hence the rank of E_2 for C_2' is equal to the number of 2-cells of C_2' if and only if the rank of E_2 for C_2 is equal to the number of 2-cells of C_2.

Since a regular subdivision of C_2 can be effected by the two operations of introducing new 0-cells on the 1-cells of C_2 and separating the 2-cells into new 2-cells by 1-cells, it follows from the theorem just proved that any regular subdivision of C_2 is such that

$$r_2 = \alpha_2 - 1$$

if and only if C_2 has this property.

59. If C_2 is a 2-circuit and G_2 is any 2-circuit homeomorphic with C_2, let K_2 be the 2-circuit on C_2 whose cells are respectively homeomorphic with the cells of G_2. As in § 50 C_2 and K_2 may be regularly subdivided into $\overline{C}_2$ and $\overline{K}_2$ and a set of spheres S_2^p constructed such that the sum (mod. 2) of $\overline{K}_2$ and the 2-circuits defining these spheres is $\overline{C}_2$. For each 2-cell k_p^2 of $\overline{K}_2$ there is one and only one sphere S_2^p which has k_p^2 as one of its 2-cells.

If K_2 is such that $r_2 = \alpha_2 - 1$, $\overline{K}_2$ has the same property, that is to say, some linear combination of the oriented boundaries of its 2-cells sums to zero. Each of the spheres S_2^p obviously has this property also. The set of oriented 1-circuits which can be formed from the boundaries of the 2-cells of $\overline{K}_2$ and of the spheres S_2^p is therefore subject to one linear relation involving the oriented 1-circuits of $\overline{K}_2$ and

one analogous linear relation for each of the spheres S_2^p. Since each S_2^p has just one 2-cell in common with $\bar{K}_2$, the linear relations corresponding to the spheres S_2^p can be multiplied by integers and added to the linear relation corresponding to $\bar{K}_2$ in such a way that all terms involving oriented 1-circuits of $\bar{K}_2$ cancel out, thus giving a linear relation, R, among oriented 1-circuits bounding 2-cells of the spheres S_2^p which does not involve any oriented 1-circuit bounding a 2-cell of $\bar{K}_2$.

Among the 2-cells of the spheres S_2^p are the 2-cells b_i^2 each determined as explained in § 41 by a 1-cell k_i^1 of $\bar{K}_2$. Each such 2-cell is in the spheres S_2^p corresponding to the 2-cells of $\bar{K}_2$ incident with the k_i^1 in question, and no others. Since the oriented circuits bounding 2-cells of $\bar{K}_2$ which are incident with k_i^1 were cancelled out in forming R, the oriented 1-circuit formed from the boundary of b_i^2 is also cancelled out. Hence R contains none of the oriented 1-circuits formed from the boundaries of the 2-cells b_i^2. Hence R can only contain oriented 1-circuits formed from the boundaries of 2-cells of $\bar{C}_2$. It must contain some of these, for otherwise each 2-cell of $\bar{C}_2$ would be in an even number of spheres S_2^p and hence the sum (mod. 2) of these spheres S_2^p and the complex $\bar{K}_2$ would be zero contrary to § 51.

Hence the set of oriented 1-circuits formed from the boundaries of the 2-cells of $\bar{C}_2$ is subject to one linear condition. Hence by § 55 $r_2 = \alpha_2 - 1$ for $\bar{C}_2$. Hence by § 58 $r_2 = \alpha_2 - 1$ for C_2.

60. The theorem of § 53 was that if C_2 is a 2-circuit any complex homeomorphic with C_2 is a 2-circuit. The theorem of the last section adds to this result the theorem that *if C_2 is orientable so is also any complex homeomorphic with C_2*. It follows that if one of the complexes into which a manifold can be decomposed is orientable so are all the complexes into which it can be decomposed. Thus the property of orientability or non-orientability is a property of a manifold and is invariant under the group of homeomorphisms.

As a corollary of this it follows that any complex defining a sphere is orientable. The same follows for any sphere

with p handles on observing that the particular complexes used in defining these manifolds are orientable. In like manner, the manifolds defined in § 26 are non-orientable.

Normal Forms for Manifolds

61. It has now been proved that any two homeomorphic manifolds are both orientable or both one-sided, and have the same connectivity. Conversely it can be proved that *if two closed manifolds are both orientable (or both one-sided) and have the same connectivity they are homeomorphic.* In other words, R_1 and the orientableness of a closed manifold characterize it completely from the point of view of Analysis Situs.

62. By way of establishing this theorem we shall outline a method of reducing any manifold to a normal form. Let C_2 denote a complex whose points constitute a manifold M_2. Let the 2-cells of C_2 be so ordered that a_2^k $(k = 2, 3, \cdots, \alpha_2)$ is incident with at least one 1-cell, say a_{k-1}^1, which is also incident with one of the 2-cells $a_1^2, a_2^2, \cdots, a_{k-1}^2$.

According to § 9 the cells a_1^2, a_1^1, a_2^2 constitute a 2-cell, b_2^2. Similarly, the cells b_2^2, a_2^1, a_3^2 constitute a 2-cell, b_3^2. The process may be continued until we arrive at a 2-cell $b_{\alpha_2}^2$ which is made up of all the 2-cells a_i^2 $(i = 1, 2, \cdots, \alpha_2)$ and of the 1-cells a_j^1 $(j = 1, 2, \cdots, \alpha_2 - 1)$. The remaining 1-cells are in number $\alpha_1 - \alpha_2 + 1 = \alpha_0 + R_1 - 2$ (§ 30, equation (4)). Hence the boundary of $b_{\alpha_2}^2$ contains $2(\alpha_0 + R_1 - 2)$ 1-cells which coincide by pairs with the 1-cells a_k^1 $(k = \alpha_2, \alpha_2 + 1, \cdots, \alpha_1)$. We denote by U_1 the linear graph determined by the 1-cells a_k^1.

63. The graph U_1 has the property that none of its 1-circuits or sets of 1-circuits bounds. For if K_1 were a bounding set of 1-circuits composed of cells of U_1, then C_2 would be separated by K_1 into two parts, each bounded by K_1. In the sequence of cells a_i^2 $(i = 1, 2, \cdots, \alpha_2)$ there must be at least one pair a_j^2, a_{j+1}^2 such that a_j^2 would be in one of the parts in question and a_{j+1}^2 would be in the other. Hence

the 1-cell a_j^1 must be on the common boundary of the two parts, namely K_1, hence on U_1. But this would contradict the definition of U_1 as a linear graph containing none of the 1-cells a_j^1 $(j = 1, 2, \cdots, \alpha_2 - 1)$. Consequently U_1 has the property stated above.

64. The result of the last section may be stated in the following form: Any closed manifold M_2 can be set into continuous correspondence with the points of a convex polygon of $2(\alpha_0 + R_1 - 2)$ edges in a Euclidean plane in such a way that (1) each interior point of the polygon corresponds to and is the correspondent of one point of the manifold; (2) each interior point of an edge of the polygon determines an interior point of another edge such that these two points of the polygon correspond to one point of the manifold, and this point of the manifold corresponds only to these two points of the polygon; (3) each vertex of the polygon determines a set of vertices of the polygon all of which correspond to a single point of the manifold, and this point of the manifold corresponds to these vertices and these only.

65. By a series of transformations on this polygon which involve cutting it by 1-cells running from one vertex to another and piecing it together along corresponding edges, it can be changed into a polygon of $2(R_1 - 1)$ sides all of whose vertices correspond to a single 0-cell of M_2. This polygon in turn can be transformed into one of three normal forms. If the polygon reduces to the first of these forms the manifold is a sphere with p handles; if the polygon takes the second form, the manifold is a one-sided manifold of the first kind; and if the polygon takes the third form, the manifold is a one-sided manifold of the second kind. Thus, every closed manifold M_2 is of one of the three types described in §§ 25 and 26.

A proof of this theorem which follows the line of argument outlined above is to be found in a paper by H. R. Brahana in the Annals of Mathematics (2), Vol. 23 (1921), pp. 144–68.

CHAPTER III

COMPLEXES AND MANIFOLDS OF n DIMENSIONS

Fundamental Definitions

1. In a Euclidean three-space, four non-coplanar points together with the one- and two-dimensional simplexes (§ 1, Chap. I and § 1, Chap. II) of which they are vertices constitute the boundary of a finite region, called a *three-dimensional simplex* or *tetrahedral region,* of which the four given points are called the *vertices.* The points of the boundary are not regarded as points of the simplex.

A set of $n+1$ points, not all in the same $(n-1)$ space, together with the one-, two-, $\cdots$, $(n-1)$-dimensional simplexes of which they are vertices constitute the boundary of a finite region in the n-space containing the $n+1$ points. This region is called an *n-dimensional simplex* and the $n+1$ given points are called its vertices. The points of the boundary are not regarded as points of the simplex.

Consider any set of objects in (1–1) correspondence with the points of an n-dimensional simplex $(n>0)$ and its boundary. The objects corresponding to the points of the simplex constitute what is called an *n-dimensional cell* or *n-cell,* and those corresponding to the boundary of the simplex what is called the *boundary of the cell.*

The remarks of § 2, Chap. I are now to be applied without change to the n-dimensional case.

2. An n-dimensional complex is defined by the following recursive statements:

An *n-dimensional complex* C_n consists of an $(n-1)$-dimensional complex C_{n-1} together with a number, α_n, of n-cells whose boundaries are circuits of C_{n-1}, such that no n-cell has a point in common with another n-cell or with C_{n-1} and such

that each $(n-1)$-cell of C_{n-1} is on the boundary of at least one n-cell. The order relations of the points of the boundary of each n-cell coincide with the order relations among these points regarded as belonging to the $(n-1)$-dimensional circuit.* The $(n-k)$-cells $(k = 1, 2, \cdots, n)$ on the boundary of an n-cell of C_n are said to be *incident* with it and it is said to be *incident* with them.

An *n-dimensional circuit* or *n-circuit* or generalized n-dimensional polyhedron is an n-dimensional complex C_n such that (1) each $(n-1)$-cell of C_n is incident with an even number of n-cells and (2) no subset of the cells which constitute C_n satisfies (1).

The definition of *homeomorphism* and the remarks in § 3, Chap. II generalize directly to n dimensions. In particular, *any theorem about an n-dimensional complex which remains valid if the complex is subjected to any* (1–1) *continuous transformation is a theorem of Analysis Situs.*

An arbitrary subset of the cells of an n-dimensional complex is sometimes referred to as a *generalized n-dimensional complex,* provided it contains at least one n-cell.

3. The definition of a singular or non-singular generalized complex C_k on a complex C_n is a direct generalization of that given in § 33, Chap. II. It is obtained from the definition in Chap. II by substituting C_k for C', C_n for C_2 and making corresponding substitutions wherever the dimensionality of cells or complexes is mentioned. The number k may be greater than, equal to, or less than n.

It is important to notice that in the fundamental definitions

* This statement can also be put in the following form: Suppose that an i-cell a^i appears on the boundaries of two $(i+k)$-cells, a_1^{i+k} and a_2^{i+k}. Then a_1^{i+k} and a_2^{i+k} and their boundaries are, by definition, in (1–1) correspondences T_1 and T_2 with two $(i+k)$-dimensional simplexes, b and c and their boundaries. In the correspondence T_1 a^i corresponds to an i-dimensional cell b^i of the boundary of b while in the correspondence T_2 it corresponds to an i-dimensional cell c^i of the boundary of c. The resultant of the correspondences effected by T_1^{-1} and T_2 on b^i and a^i respectively is a correspondence in which b^i corresponds to c^i. *This correspondence must be continuous.*

in the two sections above all the cells and the circuits bounding them are non-singular. This insures that the representation by matrices given below shall be unique. It does not, however, exclude the possibility of extending the use of the matrices to cases where, as in § 32, Chap. II, the cells have singular boundaries. But in proving our general theorems we stick to the case of non-singular cells with non-singular boundaries.

Matrices of Incidence

4. Let $\alpha_k\,(k = 0, 1, \cdots, n)$ denote the number of k-cells in a complex C_n. The k-cells themselves may be denoted by $a_1^k, a_2^k, \cdots, a_{\alpha_k}^k$. The incidence relations between the $(k-1)$-cells and the k-cells are represented by a matrix

$$\|\eta_{ij}^k\| = H_k \qquad (k = 1, 2, \cdots, n)$$

in which $\eta_{ij}^k = 1$ if a_i^{k-1} is incident with a_j^k and $\eta_{ij}^k = 0$ if a_i^{k-1} is not incident with a_j^k. The matrix H_k has α_{k-1} rows and α_k columns.

An n-dimensional complex is completely described by the set of matrices,

$$H_1, H_2, \cdots, H_n,$$

for, as can be shown by an obvious argument (cf. § 6, Chap. II) any two complexes having the same set of matrices are in (1-1) continuous correspondence.

The elements of the matrices are combined as integers reduced modulo 2, just as in Chap. I. The ranks of the matrices are denoted by $\varrho_1, \varrho_2, \cdots, \varrho_n$ respectively.

By the general theory of such matrices, there exists for each H_k a pair of square matrices A_{k-1}, B_k, of α_{k-1} and α_k rows respectively, each having its determinant equal to 1, such that

$$A_{k-1}^{-1} \cdot H_k \cdot B_k = H_k^*,$$

where H_k^* is a matrix of α_{k-1} rows and α_k columns in which the first ϱ_k elements of the main diagonal are unity and all the rest of the elements are zero. Thus the theory of the

n-dimensional complex will involve the matrices H_i, A_{i-1}, B_i, H_i^*, $(i = 1, 2, \cdots, n)$.

5. Special cases to illustrate the incidence matrices are easily constructed. For example the matrices for a complex obtained by subdividing a projective 3-space into cells are given in Chap. IX, Vol. II of the Veblen and Young Projective Geometry. The following definition gives another example.

By an *n-dimensional sphere* or a *simple closed manifold of n dimensions* is meant the set of points on a complex whose matrices of incidence are

$$H_1 = H_2 = \cdots = H_n = \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}.$$

The n-dimensional sphere is easily seen to be homeomorphic with the boundary of an $(n+1)$-cell. Since it has two 0-cells, two 1-cells, $\cdots$, two n-cells, its *characteristic*,

$$\alpha_0 - \alpha_1 + \alpha_2 - \cdots + (-1)^n \alpha_n,$$

is 0 if n is odd and 2 if n is even.

6. Any set of the k-cells, $a_1^k, a_2^k, \cdots, a_{\alpha_k}^k$, and also the k-dimensional complex consisting of a set of k-cells and their boundaries, may be denoted by a symbol $(x_1, x_2, \cdots, x_{\alpha_k})$, in which $x_i = 1$ if a_i^k is in the set and $x_i = 0$ if a_i^k is not in the set.

These symbols can be added (mod. 2) by precisely the rule given in §§ 14 and 15, Chap. I, for the 0- and 1-dimensional cases. Corresponding to this we have a rule for the addition of two k-dimensional complexes consisting each of a set of k-cells and their boundaries. The *sum*, modulo 2, of two n-dimensional complexes C_n' and C_n'' each of which is a subcomplex of a given complex C_n, is the complex determined by the set of all k-cells in C_n' or C_n'' but not in both C_n' and C_n''; it is denoted by $C_n' + C_n''$ (mod. 2). It has the obvious property that if C_n' and C_n'' are n-circuits, $C_n' + C_n''$ (mod. 2) is also an n-circuit or a set of n-circuits.

7. The *boundary* of a k-dimensional complex C_k is the $(k-1)$-dimensional complex consisting of the $(k-1)$-cells of the

complex C_n which are incident each with an odd number of k-cells of C_k, and the boundaries of these $(k-1)$-cells. Thus a k-dimensional complex is a set of k-circuits if and only if it has no boundary.

By precisely the same reasoning as that used in the 0- and 1-dimensional cases (cf. § 28, Chap. II) the boundary of a C_k is a $(k-1)$-dimensional circuit or a set of $(k-1)$-dimensional circuits having at most a $(k-2)$-dimensional complex in common. From this reasoning it also follows that every bounding $(k-1)$-circuit is a sum (mod. 2) of a set of $(k-1)$-circuits which bound k-cells, i. e., which are represented by columns of H_k. Hence all bounding $(k-1)$-circuits are linearly expressible in terms of those corresponding to a linearly independent set of ϱ_k columns of H_k, where ϱ_k is the rank of H_k.

8. As in the 0-, 1-, and 2-dimensional cases (cf. § 24, Chap. I),

$$\eta_{i1}^k x_1 + \eta_{i2}^k x_2 + \cdots + \eta_{i\alpha_k}^k x_{\alpha_k}$$

is 1 or 0 according as there are an odd or an even number of k-cells of the set $(x_1, x_2, \cdots, x_{\alpha_k})$ incident with the $(k-1)$-cell a_i^{k-1}. Hence if

$$(1) \qquad H_k \cdot \begin{Vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_k} \end{Vmatrix} = \begin{Vmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{\alpha_{k-1}} \end{Vmatrix},$$

$(y_1, y_2, \cdots, y_{\alpha_{k-1}})$ represents the boundary of $(x_1, x_2, \cdots, x_{\alpha_k})$.

As a corollary it follows that the k-circuits are the solutions of the equations

$$(H_k) \qquad \sum_{j=1}^{\alpha_k} \eta_{ij}^k x_j = 0 \qquad (i = 1, 2, \cdots, \alpha_{k-1}).$$

Since the columns of the matrix H_k represent $(k-1)$-circuits they represent solutions of the equations

(H_{k-1}) $$\sum_{j=1}^{\alpha_{k-1}} \eta_{ij}^{k-1} x_j = 0 \qquad (i = 1, 2, \cdots, \alpha_{k-2})$$

and hence

(2) $$H_{k-1} \cdot H_k = 0 \qquad (k = 1, 2, \cdots, n).$$

The Connectivities R_i

9. If ϱ_k denotes the rank of H_k (mod. 2) the number of solutions of the linear homogeneous equations (H_k) in a complete set is $\alpha_k - \varrho_k$ (cf. § 25, Chap. I). According to § 8, the columns of H_{k+1} are solutions of the equations (H_k) and hence ϱ_{k+1} of these columns can enter in a complete set of solutions of (H_k).

Let $R_k - 1$ be the smallest number of non-bounding k-circuits which it is necessary to adjoin to a set of ϱ_{k+1} linearly independent bounding k-circuits in order to have a set of k-circuits on which all others are linearly dependent.

Then for an n-dimensional complex C_n the number of solutions of (H_k) in a complete set is $\varrho_{k+1} + R_k - 1$ if $0 < k < n$. Hence

$$\alpha_k - \varrho_k = \varrho_{k+1} + R_k - 1 \qquad (0 < k < n)$$

and

$$\alpha_n - \varrho_n = R_n - 1.$$

By § 20, Chap. I

$$\alpha_0 - \varrho_1 = R_0.$$

Hence we have the series of equations

(1) $$\begin{aligned} R_0 \quad - 1 &= \alpha_0 \quad - \varrho_1 \quad - 1, \\ R_1 \quad - 1 &= \alpha_1 \quad - \varrho_1 \quad - \varrho_2, \\ R_2 \quad - 1 &= \alpha_2 \quad - \varrho_2 \quad - \varrho_3, \\ &\cdot \\ &\cdot \\ &\cdot \\ R_{n-1} - 1 &= \alpha_{n-1} - \varrho_{n-1} - \varrho_n, \\ R_n \quad - 1 &= \alpha_n \quad - \varrho_n. \end{aligned}$$

On multiplying these equations alternately by $+1$ and -1 and adding we obtain

(2) $$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + \sum_{i=0}^{n} (-1)^i (R_i - 1).$$

In case the complex C_n is an n-circuit, $R_0 = 1$, $R_n = 2$ and (2) becomes

$$(3) \qquad \sum_{i=0}^{n}(-1)^i\alpha_i = 1+(-1)^n+\sum_{i=1}^{n-1}(-1)^i(R_i-1).$$

This is a generalization of Euler's formula (§ 30, Chap. II) to n dimensions. If n is even it reduces to

$$(4) \qquad \alpha_0-\alpha_1+\alpha_2-\cdots+\alpha_n = 3-R_1+R_2-\cdots-R_{n-1}.$$

In case C_n is a manifold and n is odd, (3) when combined with a result obtained in § 29 below reduces to

$$(5) \qquad \alpha_0-\alpha_1+\alpha_2-\cdots-\alpha_n = 0.$$

10. The number $\alpha_0-\alpha_1+\cdots+(-1)^n\alpha_n$ is called the *characteristic* of the complex C_n. The number R_i $(i = 0, 1, 2, \cdots, n)$ is called the *connectivity of the ith order.*

It will presently be proved that the connectivity numbers $R_0, R_1, \cdots, R_n$ are Analysis Situs invariants. From this it will follow that the characteristic is also an invariant.

Reduction of the Matrices H_k to Normal Form

11. Let us now consider the matrices A_{k-1} and B_k by which H_k is reduced to its normal form, i. e., the square matrices of determinant 1 such that

$$(1) \qquad A_{k-1}^{-1}\cdot H_k\cdot B_k = H_k^*$$

where the first ϱ_k elements of the main diagonal of H_k^* are 1 and all the other elements of H_k^* are 0. The existence of these matrices follows from the general theory of matrices (cf. § 49, Chap. I) and we shall show that they can be so chosen as to satisfy certain additional conditions analogous to those found in §§ 30–32, Chap. I.

Writing (1) in the form

$$(2) \qquad H_k\cdot B_k = A_{k-1}\cdot H_k^*$$

it follows from § 8 that each of the first ϱ_k columns of B_k represents a k-dimensional complex bounded by the $(k-1)$-dimensional complex represented by the corresponding column of A_{k-1}. Each of the remaining $\alpha_k - \varrho_k$ columns of B_k represents a k-dimensional complex which has no boundary, i. e., a k-dimensional circuit or set of circuits.

Since B_k is a square matrix of α_k rows whose determinant is 1, every symbol of the form $(x_1, x_2, \cdots, x_{\alpha_k})$ in which the elements are reduced modulo 2 is expressible as a linear combination of the columns of B_k. Hence the symbol for any k-dimensional complex determined by k-cells of C_n is expressible in terms of the columns of B_k. Moreover since the last $\alpha_k - \varrho_k$ columns of B_k are linearly independent and the symbols for all k-circuits are linearly dependent on $\alpha_k - \varrho_k$ of them, the last $\alpha_k - \varrho_k$ columns of B_k are a complete set of k-circuits or sets of k-circuits.

Thus the reduction of the incidence matrices to normal form affords an explicit method of determining the bounding and non-bounding sets of circuits of all dimensionalities.

12. The equation (2) remains valid if we add a given column of B_k to another column of B_k and perform the corresponding operation on the columns of $A_{k-1} \cdot H_k^*$. Hence in particular we may replace any one of the last $\alpha_k - \varrho_k$ columns of B_k by any linear combination of these columns (hence by any symbol for a set of k-circuits) without modifying the right member of (2) since all the last $\alpha_k - \varrho_k$ columns of $A_{k-1} \cdot H_k^*$ are composed of zeros.

13. Suppose we change B_{k-1} by replacing its last ϱ_k columns by the first ϱ_k columns of A_{k-1}, and replacing the preceding $\alpha_{k-1} - \varrho_{k-1} - \varrho_k$ columns by the symbols for a set of $(k-1)$-circuits no combination of which bounds, the existence of which follows from § 9. By § 12 such a change will leave (2) still valid; hence to show that it is permissible it is sufficient to prove that the new B_{k-1} has determinant 1.

We now have the columns of the new B_{k-1} in three blocks, of which the first is the same as for the old B_{k-1}. The symbol for any $(k-1)$-dimensional complex is a sum of

columns of this first block and a symbol for a set of $(k-1)$-circuits, as follows from the structure of the original B_{k-1}. Now the columns of the last two blocks are linearly independent (mod. 2) as follows from their choice, and since they number $\alpha_{k-1}-\rho_{k-1}$ it follows from § 9 that the symbol for any set of $(k-1)$-circuits is a sum of these columns.

Thus the symbol for any $(k-1)$-dimensional complex is a sum of columns of the new B_{k-1}. Consequently the determinant of the new B_{k-1} must be 1 (mod. 2), and the change proposed above can be made. Let this be done for all values of k from 1 to n. The last ρ_k columns of B_{k-1} then represent bounding sets of $(k-1)$-circuits and the $R_{k-1}-1$ columns preceding these represent non-bounding $(k-1)$-circuits.

Since all rows of H_k^* after the ρ_kth contain only zeros the last $\alpha_{k-1}-\rho_k$ columns of A_{k-1} are arbitrary subject to the condition that the determinant of A_{k-1} shall be 1. Hence these columns of A_{k-1} may be taken as identical with the first $\rho_{k-1}+R_{k-1}-1$ columns of B_{k-1}. Let this be done for all values of k from 1 to n.

14. By this process it is brought about that the matrices A_k are identical with the matrices B_k except for a permutation of columns. The columns of each matrix B_k fall into three blocks. The first ρ_k columns represent single k-dimensional complexes bounded by sets of $(k-1)$-circuits. Each of the next R_k-1 columns represents a single non-bounding k-circuit. The last ρ_{k+1} columns represent bounding sets of k-circuits.

Congruences and Homologies, Modulo 2

15. The definition of congruences and homologies modulo 2 which was made in §§ 37, 38, Chap. II, applies without change to the n-dimensional case. Thus

$$C_k \equiv C_{k-1} \quad (\text{mod. } 2) \tag{1}$$

means that C_{k-1} is the boundary of C_k; and with reference to a complex C_n

$$C_{k-1} \sim 0 \quad (\text{mod. } 2) \tag{2}$$

means that there exists a complex C_k on C_n which satisfies the congruence (1). The remarks about linear combination

of congruences and complexes made in Chap. II apply here without change.

All the relations stated above by means of the matrices H_k can also be expressed in terms of congruences and homologies. For if we let a_j^k ($j = 1, 2, \cdots, \alpha_k$; $k = 1, 2, \cdots, n$) represent the cell a_j^k and its boundary, instead of the cell alone as in the notation heretofore used, we have the congruences*

$$(3) \qquad a_j^k \equiv \sum_{i=1}^{\alpha_k-1} \eta_{ij}^k \, a_i^{k-1} \pmod{2}$$

in which η_{ij}^k are the elements of the matrix H_k. These congruences, which state the incidence relations of the complex C_n, are called the *fundamental congruences (mod. 2)*.

16. If C_k is the complex represented by $(x_1, x_2, \cdots, x_{\alpha_k})$ and C_{k-1} the set of $(k-1)$-circuits represented by $(y_1, y_2, \cdots, y_{\alpha_{k-1}})$, the congruence (1) is equivalent to the matrix equation (1) of § 8. The result of reducing the incidence matrices to normal form as summarized in § 14 therefore amounts to the statement that the fundamental congruences are equivalent to the following set of congruences and homologies

$$(4) \qquad \left.\begin{array}{ll} K_k^1 & \equiv C_{k-1}^{R_{k-1}} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ K_k^{\rho_k} & \equiv C_{k-1}^{R_{k-1}+\rho_k-1} \\ C_k^1 & \equiv 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ C_k^{R_k-1} & \equiv 0 \\ C_k^{R_k} & \sim 0 \\ \cdot & \\ \cdot & \\ \cdot & \\ C_k^{R_k+\rho_{k+1}-1} & \sim 0. \end{array}\right\} \pmod{2}$$

* We are here making the obvious convention that $\eta a_i^{k-1} = a_i^{k-1}$ if $\eta = 1$ and $\eta a_i^{k-1} = 0$ if $\eta = 0$.

The further study of these congruences and homologies will involve proving (1) that the k-circuits $C_k^1, C_k^2, \cdots, C_k^{R_k-1}$ are not homologous to zero (mod. 2) and (2) that every k-circuit on C_n is homologous to a combination of them. With regard to the statement (1) the discussion up to the present shows that no combination of these k-circuits bounds any complex composed of cells of C_n. And with regard to (2) we know that every k-circuit composed of cells of C_n is homologous to a combination of $C_k^1, C_k^2, \cdots, C_k^{R_k-1}$. To bring complexes on C_n which are not composed of cells of C_n into consideration it will be necessary to go beyond the combinatorial properties of C_n and make use of the geometrical properties of the cells.

Theory of the n-Cell

17. The combinatorial properties of a complex C_n which have been discussed above have an elementary application in the theory of the subdivision of a Euclidean space by generalized polyhedra. A system of $(n-1)$-spaces in an n-space subdivide the n-space into a set of n-dimensional convex regions. They intersect in a number of $(n-2)$-spaces which subdivide each $(n-1)$-space into a set of $(n-1)$-dimensional convex regions which bound the n-dimensional convex regions. The $(n-2)$-spaces have $(n-3)$-spaces in common which divide the $(n-2)$-spaces into convex regions, and so on. Thus the set of $(n-1)$-spaces defines a subdivision of the n-space into a set of cells which can be treated by the methods described above. Any k-circuit formed from the k-dimensional convex regions is a generalized polyhedron. Any such k-circuit bounds a $(k+1)$-dimensional complex composed of convex $(k+1)$-cells.

A treatment of the theory of polyhedra from this point of view by the author is to be found in the Transactions of the American Math. Soc., Vol. 14 (1913), p. 65. (See also the correction Vol. 15, p. 506.) Earlier and later treatments without the machinery used here are to be found in the papers by N. J. Lennes, Am. Journ. of Math., Vol. 33 (1911),

p. 37, and Lilly Hahn, Monatshefte für Math. u. Phys., Vol. 25 (1914), p. 303. Since an n-cell is homeomorphic with a Euclidean space all this is the most elementary part of the theory of the n-cell.

18. As in § 8, Chap. II, we can define a system of curves in any n-cell $a_i^n (i = 1, 2, \cdots, \alpha_n)$ which have the properties of the system of straight lines interior to a simplex in a Euclidean space. It is only necessary to set up a (1–1) continuous correspondence F_i between the interior and boundary of the n-cell and the interior and boundary of a simplex and to regard as *straight* those curves in the n-cell which are images of straight lines in the simplex.

Under these definitions any two points of an n-cell or its boundary determine a straight 1-cell joining them; any three non-collinear points determine a straight 2-cell bounded by them and the three straight 1-cells which they determine by pairs; in general, any $i+1$ points $(i = 1, 2, \cdots, n)$ determine a straight i-dimensional simplex bounded by the straight j dimensional simplexes $(j = 0, 1, 2, \cdots, i)$ determined by subsets of the i points.

19. From the separation theorems on Euclidean polyhedra (§ 17) there follow at once the following important corollaries, which are all to be understood as referring to complexes composed of "straight" cells:

If S_{n-2} is an $(n-2)$-dimensional sphere on the boundary of an n-cell a^n the boundary of a^n consists of S_{n-2} and two $(n-1)$-cells a_1^{n-1} and a_2^{n-1}. Any $(n-1)$-cell a_3^{n-1} contained in a^n and bounded by S_{n-2} separates a^n into two n-cells, one bounded by a_1^{n-1}, S_{n-2}, and a_3^{n-1} and the other bounded by a_2^{n-1}, S_{n-2}, and a_3^{n-1}. There are an infinity of non-singular $(n-1)$-cells contained in a^n and bounded by S_{n-2}.

If two n-cells a_1^n, a_2^n are incident with an $(n-1)$-cell a^{n-1} and have no common point they and a^{n-1} constitute an n-cell b^n. If their boundaries have nothing in common except a^{n-1} and its boundary the boundary of b^n is the sum (mod. 2) of their boundaries.

This proposition is a special case of the following theorem: If a set of n-cells, $(n+1)$-cells, $\cdots$, $(n+p)$-cells are all incident with an $(n-1)$-cell a^{n-1} and are such that the incidence relations between the $(n+i)$-cells ($i = 0, 1, 2, \cdots, p-1$) and the $(n+i+1)$ cells are the same as those between the i-cells and $(i+1)$-cells of a p-dimensional sphere, the set of all points on a^{n-1} and the cells incident with it constitute an $(n+p)$-cell.

The set of all cells of a complex C_n which are incident with an i-cell a^i and of higher dimensionality than a^i constitute, with a^i itself, what is called a *star of cells.* If the incidence relations among the cells of a star satisfy the conditions described in the paragraph above the star is said to be *simply connected.* If a^{i+p} is one cell of a star, a^{i+p} and all cells of the star of dimensionality greater than $i+p$ which are incident with a^{i+p} constitute a star of cells.

These theorems all remain valid if the restriction to straight cells is dropped. In this more general form they depend on the generalizations to n dimensions of the Jordan and Schoenflies theorems quoted in § 10, Chap. II. The generalized Jordan theorem has been proved by L. E. J. Brouwer, Math. Ann., Vol. 71 (1911), p. 37 but the generalized Schoenflies theorem is still unproved. As in the two-dimensional case, we shall get along with the restricted form of these theorems.

Regular Complexes

20. Just as in Chap. II it was found convenient to decompose a complex into generalized triangles, here it will be found convenient to consider complexes whose n-cells are generalized simplexes. A complex is said to be *regular* if (1) each n-cell a_j^n is in such a (1–1) continuous correspondence with a simplex that each 0-cell incident with a_j^n corresponds to a vertex of the simplex, each 1-cell incident with a_j^n to an edge of the simplex and in general each i-cell ($i = 1, 2, \cdots, n-1$) incident with a_j^n corresponds to an i-dimensional simplex of the boundary of the simplex and (2) no set of $i+1$ 0-cells are the vertices of more than one i-cell of the complex.

It has been shown in Chap. II how to decompose any 2-dimensional complex C_2 into a regular complex $\overline{C}_2$. This process will now be generalized as follows:

For convenience in phraseology, let a definition of straightness be introduced for all the 2-cells of C_n in the fashion of § 18. Then let a definition of straightness be introduced for all the 3-cells, which definition may be entirely unrelated to the one used for the 2-cells. And in general let a definition of straightness be introduced for each i-cell ($i = 2, 3, \cdots, n$) quite independently of that used for all other cells.

Let $P_j^0 = a_j^0$ ($j = 1, 2, \cdots, \alpha_0$) and let P_j^i be an arbitrary point interior to the cell a_j^i ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, \alpha_i$). The points P_j^i ($i = 0, 1, 2, \cdots, n$; $j = 1, 2, \cdots, \alpha_i$) are the vertices of $\overline{C}_n$. The 1-cells of $\overline{C}_n$ are the straight 1-cells joining every point P_j^i ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, \alpha_i$) to every vertex of $\overline{C}_n$ on the boundary of a_j^i. A 2-cell of $\overline{C}_n$ is the set of points on all straight 1-cells joining a point P_j^i ($i = 2, 3, \cdots, n$; $j = 1, 2, \cdots, \alpha_i$) to the points of a 1-cell of $\overline{C}_n$ on the boundary of a_j^i. Each of these 2-cells is bounded by just three 1-cells of $\overline{C}_n$.

Continuing this process step by step we obtain the 3-cells, 4-cells, $\cdots$, n-cells of $\overline{C}_n$. A k-cell of $\overline{C}_n$ is the set of points on all straight 1-cells joining a point P_j^i ($i = k, k+1, \cdots, n$; $j = 1, 2, \cdots, \alpha_i$) to the points of a $(k-1)$-cell of $\overline{C}_n$ on the boundary of a_j^i. Each k-cell so defined is evidently bounded by $k+1$ $(k-1)$-cells.

The complex $\overline{C}_n$ thus defined is called a *regular subdivision* of C_n.

21. No two 0-cells of $\overline{C}_n$ are joined by more than one 1-cell. Hence any 1-cell of $\overline{C}_n$ may be denoted by $P_k^i P_l^j$ ($i < j$). In like manner no m 0-cells ($2 < m \leqq n+1$) are vertices of more than one $(m-1)$-cell of $\overline{C}_n$. Hence any such cell may be denoted by its vertices $P_q^i P_r^j \cdots P_v^s$. These vertices are by construction all on cells of C_n of different dimensionality. Hence they may always be taken in such an order that $i < j < \cdots < s$.

Incidentally it may be remarked here that on account of the properties just referred to, $\bar{C}_n$ may be described by means of a matrix giving the incidence relations between its n-cells and 0-cells. Also, it can be set into (1–1) continuous correspondence with a set of cells of a simplex in a Euclidean space of a sufficiently high number of dimensions. For these propositions, see the Annals of Mathematics, Vol. 14 (1913), pp. 175–177. The correspondence with cells of a Euclidean simplex can be used to introduce such a definition of distance and straightness in $\bar{C}_n$ that the straightness and distance of any cell is in agreement with the straightness and distance of any cell with which it is incident.

22. The relationship between the complexes C_n and $\bar{C}_n$ may be stated as follows:

(1) Each n-cell of C_n, a_i^n, is the sum (mod. 2) of all n-cells $P_a^0 P_b^1 \cdots P_i^n$ of $\bar{C}_n$ having P_i^n as a vertex.*

.

.

.

$(n-k+1)$ Each k-cell of C_n, a_i^k, is the sum (mod. 2) of all k-cells $P_a^0 P_b^1 \cdots P_i^k$ of $\bar{C}_n$ which have P_i^k as a vertex (the superscripts are all less than or equal to k).

.

.

.

$(n+1)$ Each 0-cell of C_n, a_i^0 is the 0-cell P_i^0.

23. *The values of $R_1, R_2, \cdots, R_n$ determined from C_n are the same as those determined from $\bar{C}_n$.* In order to prove this, consider any i-circuit K_i of $\bar{C}_n$ which is not simply a subdivision of an i-circuit of C_n, and which therefore contains at least one of the points P_j^m, $m > i$. We choose such a point for which m has its maximum value. The i-cells of K^i which are incident with P_j^m are then incident with $(i-1)$-cells of the boundary of the cell a_j^m of C_n. These $(i-1)$-cells of the boundary of a_j^m constitute one or more $(i-1)$-circuits K_{i-1}^a because the $(i-1)$-cells of K_i which are incident with

* The sum (mod. 2) of a set of k-cells of a star will be understood to contain the cells of the star that are on the boundaries of the k-cells of the sum.

P_j^m and with $(i-2)$-cells of the boundary of a_j^m are incident each with an even number of i-cells of K_i. Now by mathematical induction we may assume the invariance of the connectivity numbers for dimensions less than n, since we shall later establish it for the dimension n (§ 42). Hence K_{i-1}^a bounds at least one i-dimensional complex C_i^a composed of cells of $\bar{C}_n$ on the boundary of a_j^m. By its definition it also bounds a complex composed of i-cells of K_i which are incident with P_j^m. These two complexes constitute an i-circuit or set of i-circuits K_i^a, which bounds the complex composed of the $(i+1)$-cells of $\bar{C}_n$ which are incident with P_j^m and the i-cells of C_i^a. If K_i^a is added (mod. 2) to K_i the resulting set of i-circuits K_i' does not pass through P_j^m. Repeating this argument until there are no longer any vertices P_j^m, $m>i$, of $\bar{C}_n$ on K_i, it follows that by adding bounding sets of circuits to K_i it can be converted into a set of i-circuits which does not pass through any of the vertices P_k^m, $m>i$, of $\bar{C}_n$. Such a set of i-circuits is simply a subdivision of a set of i-circuits of C_n.

From this it follows that all i-circuits of $\bar{C}_n$ are linearly dependent on bounding sets of circuits and circuits coincident with circuits of C_n. Hence the value of R_i determined by $\bar{C}_n$ is not greater than that determined by C_n. It also cannot be less, for if so there would be a linear relation among the i-circuits C_i^p $(p=1, 2, \cdots, R_i-1)$ regarded as circuits of $\bar{C}_n$. But this would mean that there was a complex K_{i+1} composed of cells of $\bar{C}_n$ and bounded by some or all of the circuits C_i^p. By an argument like that in the paragraph above K_{i+1} could be replaced by a complex K_{i+1}' coincident with a complex composed of cells of C_n. But the existence of K_{i+1}' would mean a linear relation among the i-circuits C_i^p regarded as i-circuits of C_n. Hence the value of R_i determined by $\bar{C}_n$ is not less than that determined by C_n.

Manifolds

24. By a *neighborhood* of any i-cell a^i on a complex C_n is meant any set S of non-singular cells on C_n such that any

set of points of C_n having a limit point on a^i contains points on the cells of S.

If C_n is an n-circuit such that every star of its cells is simply connected, the set of points on C_n is called a closed *n-dimensional manifold*. It is easily proved that any regular subdivision of such a C_n satisfies the same conditions. This definition implies that every point of a manifold has a neighborhood which is an n-cell. It has not been proved, however, that if a point set satisfies the above conditions for one subdivision into cells, it satisfies them for all other subdivisions into cells.

Dual Complexes

25. A complex C'_n is said to be *dual* to a complex C_n if the incidence relations between the k-cells and $(k-1)$-cells of C'_n are the same as those between the $(n-k)$-cells and $(n-k+1)$-cells of C_n for $k = 1, 2, \cdots, n$. In case C_n defines a manifold, a complex C'_n dual to C_n can be constructed by first making a regular subdivision of C_n into $\overline{C}_n$, then defining as an n-cell of C'_n the set of all points on each star of cells of $\overline{C}_n$ having a vertex of C_n as center, next defining as an $(n-1)$-cell of C'_n the set of all points on each star of cells of dimensionality $n-1$ and less which are incident with the point P^1_i on a 1-cell of C_n, but are not incident with any P^0_j, and so on, finally defining as the 0-cells of C'_n the points P^n_i on the n-cells of C_n.

This process is illustrated in Fig. 3, page 44 for the two-dimensional case. In this figure the vertices of C'_2 are the points P^2_i, the 1-cells of C'_2 are made up of the pairs of 1-cells $P^1_i P^2_j$, $P^1_i P^2_k$ of $\overline{C}_2$, and the 2-cells of C'_2 are the triangle stars at the vertices of C_2.

26. The construction for C'_n may be stated a little more explicitly in terms of our notations (cf. § 22) as follows:

(1) Each 0-cell of C'_n is the 0-cell P^n_i.

$(n-k+1)$ Each $(n-k)$-cell of C'_n, b_i^{n-k}, is the sum (mod. 2) of all $(n-k)$-cells $P_i^k P_j^{k+1} \cdots P_p^n$ of $\overline{C}_n$ which have P_i^k as a vertex.

.

.

.

$(n+1)$ Each n-cell of C'_n, b_i^n, is the sum (mod. 2) of all n-cells $P_i^0 P_j^1 \cdots P_t^n$ of $\overline{C}_n$ which have P_i^0 as a vertex.

In order to make sure that this actually defines a complex dual to C_n it must be proved first that each of the statements $(1) \cdots (n+1)$ defines a cell and second that the set of cells has the properties required of a dual complex.

27. Consider first the statement $(n+1)$. The 0-cell P_i^0 is a vertex a_i^0 of C_n. Since we are dealing with a manifold, a_i^0 and the set of all cells of $\overline{C}_n$ incident with it form a simply connected star, and the set of points on this star form an n-cell. This n-cell we have called b_i^n.

No two of the n-cells b_i^n have a point in common because no n-cell of $\overline{C}_n$ is incident with more than one vertex of C_n (in the notation $P_i^0 P_j^1 \cdots P_q^n$ only one superscript is zero). Moreover every point on a cell of $\overline{C}_n$ is on the interior or boundary of one of the cells b_i^n because each n-cell of $\overline{C}_n$ is incident with at least one vertex of C_n (the superscript zero always appears once in the notation $P_i^0 P_j^1 \cdots P_q^n$).

Next consider the statement $(n-k+1)$. The point P_i^k is on the k-cell a_i^k of C_n and this k-cell contains a k-cell $P_a^0 P_b^1 \cdots P_i^k$ of $\overline{C}_n$. Since $\overline{C}_n$ is a regular subdivision of C_n, $P_a^0 P_b^1 \cdots P_i^k$ and the set of all cells of $\overline{C}_n$ of dimensionality $k+1$ or greater which are incident with it form a simply connected star (§§ 19, 24); and the set of all points on the cells of the star forms a single cell which is the sum (mod. 2) of the n-dimensional cells of the star. The n-dimensional cells of the star are all n-cells of $\overline{C}_n$ which can be denoted by $P_a^0 P_b^1 \cdots P_i^k P_j^{k+1} \cdots P_p^n$ in which the first $k+1$ of the P's are fixed and the rest are variable. The incidence relations among the cells of this star are by §§ 19, 24 those of an $(n-k-1)$-dimensional sphere. These incidence relations

are however the same as those among the $(n-k)$-cells $P_i^k P_j^{k+1} \cdots P_p^n$ described in the statement $(n-k+1)$ and the cells of lower dimensionality with which they are incident. Hence the sum, (mod. 2) of the cells $P_i^k P_j^{k+1} \cdots P_p^n$ described in the statement $(n-k+1)$ is an $(n-k)$-cell. This $(n-k)$-cell we call b_i^{n-k}. It obviously has the point P_i^k, and this point only, in common with a_i^k.

28. Let us next find the incidence relations among the b's. If a_i^k is incident with a_j^{k+1}, there is a k-cell, $P_a^0 P_b^1 \cdots P_i^k$, of $\overline{C}_n$ contained in a_i^k which is incident with the $(k+1)$-cell, $P_a^0 P_b^1 \cdots P_i^k P_j^{k+1}$, contained in a_j^{k+1}. The cell b_i^{n-k} dual to a_i^k is the sum (mod. 2) of all the $(n-k)$-cells $P_i^k P_j^{k+1} P_l^{k+2} \cdots P_s^n$ for the given value of i. The cell b_j^{n-k-1} dual to a_j^{k+1} is the sum (mod. 2) of all the $(n-k-1)$-cells $P_j^{k+1} P_l^{k+2} \cdots P_s^n$ for the given value of j. Since each of the $(n-k-1)$-cells of $\overline{C}_n$ which enter into b_j^{n-k-1} is incident with an $(n-k)$-cell of $\overline{C}_n$ contained in b_i^{n-k} it follows that b_i^{n-k} is incident with b_j^{n-k-1}.

Hence if a_i^k is incident with a_j^{k+1}, b_i^{n-k} is incident with b_j^{n-k-1} The converse proposition is proved in exactly the same way. Hence a_i^k is incident with a_j^{k+1} if and only if b_i^{n-k} is incident with b_j^{n-k-1}.

Duality of the Connectivities R_i

29. Stating this result for the case $k=n-1$, we have that a_i^{n-1} is incident with a_j^n if and only if b_i^1 is incident with b_j^0. Hence the matrix of incidence relations between the 0-cells and 1-cells of the complex C_n' is the matrix H_n' obtained from the matrix H_n of the complex C_n by interchanging rows and columns. In like manner it is seen that, in general, the matrix of incidence relations between the $(n-k-1)$-cells and $(n-k)$-cells of the complex C_n' is the transposed matrix H_{k+1}' of the matrix H_{k+1} of the complex C_n. Hence the matrices of incidence $H_1, H_2, \cdots, H_n$ of C_n' are the matrices $H_n', H_{n-1}', \cdots, H_1'$ of C_n.

The ranks of these matrices are $\rho_n, \rho_{n-1}, \cdots, \rho_1$ respectively. Moreover the numbers of 0-cells, 1-cells, $\cdots$ n-cells of C_n' are $\alpha_n, \alpha_{n-1}, \cdots, \alpha_1, \alpha_0$ respectively. Hence by the formula for the i-dimensional connectivity R_i, it follows that the 1-, $\cdots$, $(n-1)$-dimensional connectivities of C_n' are $R_{n-1}, \cdots, R_1$ respectively.

It was shown in § 23 that the connectivity R_i of a complex $\bar{C}_n$ obtained by a regular subdivision of C_n is the same as that of C_n. But by comparing § 22 with § 26 it is seen that $\bar{C}_n$ is a regular subdivision both of C_n and of C_n'. Hence the connectivity R_i of C_n' is the same as that of C_n. Hence $R_{n-1}, R_{n-2}, \cdots, R_1$ are the same as $R_1, R_2, \cdots, R_{n-1}$, respectively. That is

$$R_{n-k} = R_k \qquad (k = 1, 2, \cdots, n-1).$$

It should be noted that this duality relation does not apply to R_0 and R_n. In the case of a manifold, which we are considering here, $R_0 = 1$ and $R_n = 2$.

30. An important corollary of this result is that for a manifold of an odd number of dimensions the characteristic is zero. For the equations

$$\alpha_0 - \alpha_1 + \cdots + (-1)^n \alpha_n = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (R_i - 1)$$

and

$$R_i = R_{n-i} \qquad (i = 1, 2, \cdots, n-1)$$

give

$$\alpha_0 - \alpha_1 + \alpha_2 - \cdots - \alpha_n = 0,$$

as already noted in § 9.

Generalized Manifolds

31. The definition of a manifold in § 24 can be generalized as follows: A *generalized manifold* of n dimensions is the set of all points on an n-circuit C_n such that if a^{i-1} is any cell of C_n the incidence relations among the (i)-cells, $(i+1)$-cells, $\cdots$, $(i+k)$-cells (where $i+k=n$) incident with a^{i-1} are the same as the incidence relations among the 0-cells, 1-cells, $\cdots$, k-cells of a complex defining a generalized manifold of k dimensions; a generalized manifold of zero dimensions is a 0-circuit.

For $n = 0, 1, 2$, a generalized manifold is the same as a manifold. But for $n \geqq 3$ it includes sets of points which are not manifolds in the narrow sense.

32. To bring this out let us consider the following example given in the article on Analysis Situs by Dehn and Heegaard in the Encyclopädie. Let S_4 be a Euclidean space of four dimensions, a^0 a point in S_4, S_3 a three-space in S_4 but not on a^0, and M_2 an arbitrary two-dimensional manifold (e. g., an anchor ring) in S_3. Let M_2 be decomposed into 0-cells, 1-cells and 2-cells constituting a two-dimensional complex, B_2. The segment joining any 0-cell of B_2 to a^0 is a 1-cell, the points on the segments joining the points of a 1-cell of B_2 to a^0 constitute a 2-cell, and the points on the segments joining the points of a 2-cell of B_2 to a^0 constitute a 3-cell. The complex C_3 composed of all the 1-cells, 2-cells and 3-cells formed by this process, together with a^0 and the cells of B_2, is such that the boundary of an arbitrarily small neighborhood of a^0 is of the same structure as B_2. Hence the set of points on each such boundary is a surface like M_2 (e. g., an anchor ring).

It is obvious that a generalized three-dimensional manifold can be constructed which has any number of points with neighborhoods which are not spherical. A generalized four-dimensional manifold can have both 0-cells and 1-cells whose neighborhoods are not simply connected, and so on.

33. It was shown in Chap. II that any 2-circuit can be regarded as a singular manifold. The generalization of this theorem is that any n-circuit is a *singular* (cf. § 3) *generalized manifold.* We shall repeat the process of § **34**, Chap. II, for the three-dimensional case, because one new point enters, but shall leave the formal generalization to the reader.

Let C_3 be an arbitrary 3-circuit. Each of its 2-cells a_i^2 is incident with an even number $2n_i$ of 3-cells. These may be grouped in n_i pairs of 3-cells associated with a_i^2, and the method used in § 34, Chap. II, may be used to obtain a 3-circuit C_3' whose cells coincide with those of C_3 and which

is such that each of its 2-cells is incident with two and only two of its 3-cells.

The incidence relations between the 2-cells and 3-cells of C_3' which are incident with a 1-cell a_j^1 of C_3' are the same as those of a linear graph in which each 0-cell is incident with just two 1-cells. Since such a linear graph is a set of 1-circuits having no points in common, the 2-cells and 3-cells incident with a_j^1 fall into a number, n_j, of *groups associated with* a_j^1 such that the incidence relations among the cells of a group are those of a 1-circuit. With the aid of these groups, by the method of § 34, Chap. II, a complex C_3'' is defined whose cells coincide with those of C_3' and which is such that all of its cells of dimensionality greater than i which are incident with any one of its i-cells ($i = 2, 1$) are related among themselves by a set of incidence relations identical with those of a $(2-i)$-circuit.

The incidence relations between the 1-cells, 2-cells and 3-cells incident with a 0-cell a_k^0 of C_3'' now satisfy the same conditions as those between the 0-cells, 1-cells and 2-cells of a number, n_k, of two-dimensional manifolds which have no points in common. Hence they fall into n_k *groups associated with* a_k^0 such that the incidence relations among the 1-cells, 2-cells and 3-cells of a group are the same as those among the 0-cells, 1-cells and 2-cells of a two-dimensional manifold. Hence a complex C_3''' can be defined whose cells coincide with those of C_3'' and which satisfies the definition of a generalized manifold.

C_3''' will be a manifold in the narrow sense only in the case where each of the groups associated with each vertex a_k^0 has the incidence relations of the cells of a sphere.

34. Since the boundary of any complex consists of one or more circuits, it consists of one or more generalized manifolds any or all of which may be singular.

Bounding and Non-bounding Sets of k-Circuits

35. Let us now take up the problem: Given a set of k-circuits C_k on a complex C_n, *to determine whether or not*

there exists a $(k+1)$*-dimensional complex, singular or not, on* C_n *which is bounded by* C_k. This is the problem solved in Chap. II (cf. § 35) for the case where $n=2$ and $k=1$. As the problem is now formulated k may be less than, equal to, or greater than n, and C_k may have singularities of any degree of complexity compatible with the definition in § 3.

The solution of the problem in the simplest case is contained in the following obvious theorem which is a direct generalization of that given in § 36, Chap. II: *Any sphere of k dimensions on an n-cell* a^n *is the boundary of a* $(k+1)$*-cell on* a^n. The $(k+1)$-cell can be constructed by joining an arbitrary point, P, of a^n to all the points of the k-dimensional sphere by straight 1-cells or, in case of points of the sphere which coincide with P, by singular 1-cells coincident with P. The solution of our problem for the general case which we shall now develop is entirely parallel to that carried out in §§ 39 to 46, Chap. II.

36. Let K_i be an i-dimensional complex on C_n. Let $\bar{C}_n$ be a regular sub-division of C_n. Let a definition of distance and straightness be introduced relative to $\bar{C}_n$ and let all references to distance and straightness in the rest of this argument be understood to refer to this definition. Let $\bar{\bar{C}}_n$ be a regular subdivision of $\bar{C}_n$. By simple continuity considerations it can be proved that K_i can be regularly subdivided into a complex $\bar{K}_i$ such that for each j-cell of $\bar{K}_i$ there is a star of cells of $\bar{\bar{C}}_n$ to which it is interior. *A correspondence A* is now defined as a correspondence between the vertices of $\bar{K}_i$ and those of $\bar{C}_n$ by which each vertex of $\bar{K}_i$ which is interior to a star of cells of $\bar{\bar{C}}_n$ having a vertex of $\bar{C}_n$ as center corresponds to that vertex of $\bar{C}_n$, and by which each vertex of $\bar{K}_i$ which is on the boundary of two or more stars of cells of $\bar{\bar{C}}_n$ having vertices of $\bar{C}_n$ as centers corresponds to one of these vertices of $\bar{C}_n$.

Since every point of C_n is on or on the boundary of some star of cells of $\bar{\bar{C}}_n$ with center at a vertex of $\bar{C}_n$, a correspondence A determines a unique vertex of $\bar{C}_n$ for each vertex of $\bar{K}_i$. Moreover since any cell of $\bar{K}_i$ is on a star of cells

of $\bar{C}_n$ *its vertices correspond to vertices of a single cell of* $\bar{C}_n$. Hence the correspondence A makes each cell of $\bar{K}_i$ correspond to a cell of $\bar{C}_n$ of the same or lower dimensionality.

37. Let the r-cells of $\bar{C}_n$ be denoted by $c_j^r (r = 0, 1, 2, \cdots, n; j = 1, 2, \cdots, \alpha_r)$ and those of $\bar{K}_i$ by $k_j^r (r = 0, 1, 2, \cdots, i; j = 1, 2, \cdots, \beta_r)$. Each 0-cell k_j^0 of $\bar{K}_i$ can be joined to the 0-cell of $\bar{C}_n$ to which it corresponds under the correspondence A by a straight 1-cell b_j^1; or, if k_j^0 coincides with the point to which it corresponds, by a singular 1-cell b_j^1 coinciding with k_j^0. Similarly, for each 1-cell k_j^1 of $\bar{K}_i$, a 2-cell b_j^2 can be constructed by joining each point of k_j^1 to a point of the corresponding cell of $\bar{C}_n$ by a 1-cell which is either straight or coincident with a point. By a similar construction there is determined for every cell k_j^r of $\bar{K}_i$ a cell b_j^{r+1} composed of 1-cells joining points of k_j^r to points of the cell of $\bar{C}_n$ to which k_j^r corresponds under the correspondence A. The $(i+1)$-dimensional complex composed of the cells b_j^{i+1} and their boundaries is denoted by B_{i+1}. It is such that the incidence relations of b_p^{r+1} and b_q^r are the same as those of k_p^r and k_q^{r-1}.

38. If K_i is a set of i-circuits, all i-cells $b_j^i (j = 1, 2, \cdots, \beta_{i-1})$ must cancel out when the boundaries of the $(i+1)$-cells b_j^{i+1} $(j = 1, 2, \cdots, \beta_i)$ are added together (mod. 2). Hence the boundary of B_{i+1} consists either of $\bar{K}_i$ alone or of $\bar{K}_i$ and a set of i-circuits K_i' composed of cells of $\bar{C}_n$. That is to say

$$B_{i+1} \equiv \bar{K}_i + K_i' \quad \text{(mod. 2)} \tag{1}$$

and

$$\bar{K}_i \sim K_i' \quad \text{(mod. 2)}$$

where K_i' is either zero or a set of i-circuits composed of cells of $\bar{C}_n$.

There is no difficulty in seeing that any i-circuit is homologous (mod. 2) to any regular sub-division of itself. This may be proved by means of a singular $(i+1)$-dimensional complex which contains, besides the cells of the given i-circuit and those of its subdivision, one $(k+1)$-cell incident with each k-cell of the i-circuit, $k = 0, 1, \cdots. i$. Hence

$$K_i \sim \bar{K}_i$$

and therefore

(2) $$K_i \sim K_i'.$$

It is obvious that $K_i' = 0$ if $i > n$. Hence

(3) $$K_{n+r} \sim 0 \quad (\text{mod. } 2)$$

whenever $r > 0$.

39. From the homology (2) it follows that $K_i \sim 0$ if and only if $K_i' \sim 0$. By § 7, K_i' bounds a complex composed of cells of $\bar{C}_n$ if and only if it is represented by a symbol $(x_1, x_2, \cdots, x_{\alpha_i})$ which is linearly dependent on the columns of the matrix H_{i+1} for $\bar{C}_n$. We shall now prove that if $K_i' \sim 0$, K_i' bounds a complex composed of cells of $\bar{C}_n$, from which result it obviously follows that $K_i \sim 0$ if and only if the symbol $(x_1, x_2, \cdots, x_{\alpha_i})$ for K_i' is linearly dependent on the columns of H_{i+1}.

40. Given that $K_i' \sim 0$ and that K_i' is composed of cells of $\bar{C}_n$, let K_{i+1}' be a bounded complex, and let us subdivide K_{i+1}' as above, preparatory to setting up a correspondence A. We denote the subdivision by K_{i+1}'', and the corresponding subdivision of K_i' by K_i''. Then we will have

(4) $$K_{i+1}'' \equiv K_i'' \quad (\text{mod. } 2).$$

Let us construct a correspondence A for K_{i+1}'' exactly as in § 36, and by means of it construct a complex B_{i+2} analogous to the complex B_{i+1} of § 37. When the boundaries of the $(i+2)$-cells of B_{i+2} are added to K_{i+1}'' (mod. 2), all the $(i+1)$-cells of B_{i+2} cancel except those determined by the cells k_j^i of the boundary of K_{i+1}'' and certain others which are cells of $\bar{C}_n$. Let us denote the $(i+1)$-dimensional complexes determined by these two sets of $(i+1)$-cells, by G_{i+1} and $\bar{K}_{i+1}$ respectively. This gives the congruence

(5) $$B_{i+2} \equiv K_{i+1}'' + G_{i+1} + \bar{K}_{i+1} \quad (\text{mod. } 2),$$

which implies the congruence

(6) $$K''_{i+1}+G_{i+1}+\bar{K}_{i+1}\equiv 0 \pmod{2}.$$

Since K''_i is a set of i-circuits, none of the cells of B_{i+2} determined by $(i-1)$-cells k_j^{i-1} will appear in the boundary of G_{i+1}. Hence we have

(7) $$G_{i+1}\equiv K''_i+K'''_i \pmod{2},$$

where K'''_i is a set of i-circuits composed of cells of $\bar{C}_n$. On adding (4), (6) and (7) we obtain

(8) $$K_{i+1}\equiv K'''_i \pmod{2}.$$

Hence the theorem stated in § 39 will be proved if we show that K'''_i is identical with K'_i.

To prove this, let us consider a single i-cell, say c_j^i, of K'_i. The vertices of K''_i on c_j^i or on its boundary are all assigned to vertices of c_j^i under the correspondence A. Hence the i-cells, say c_{jk}^i, of K''_i into which c_j^i is subdivided all contribute either nothing or c_j^i itself to the set of i-cells of K'''_i. Now the sum (mod. 2) of the cells identical with c_j^i obtained from the cells c_{jk}^i is bounded by whatever we get by the process applied in the previous paragraphs (to obtain K'''_i from K''_i) when that process is applied to the boundary of c_j^i as subdivided for K''_i. For that is exactly what we did prove at the end of the last paragraph, with i replaced by $i+1$, and the subdivision of c_j^i replaced by K''_{i+1}. But by mathematical induction we may assume that the boundary thus obtained is exactly the boundary of c_j^i as composed of cells of K'_i. Hence the sum (mod. 2) of the i-cells identical with c_j^i obtained as just described, being bounded by the boundary of c_j^i, must be exactly c_j^i taken once (mod. 2). Since, then, from the subdivision of every i-cell of K'_i we obtain that same cell of K'_i, from the entire subdivision, K''_i, of K'_i we must obtain K'_i in its entirety. In other words, K'''_i is identical with K'_i, which is what we set out to prove.

41. We now have an explicit method for determining whether a set of i-circuits K_i on C_n does or does not bound. For

a construction has been given to determine the homology (2) of § 38 and $K_i \sim 0$ if and only if K_i' bounds a complex composed of cells of $\bar{C}_n$.

It is a corollary that no set of n-circuits composed of cells of C_n can satisfy a homology $K_n \sim 0$. For there are no $(n+1)$-cells in $\bar{C}_n$. Hence, in particular, a set of n-circuits C_n cannot bound a singular complex on C_n. On the other hand, every $(n+k)$-circuit $(k>0)$ on C_n bounds an $(n+k+1)$-dimensional complex on C_n as stated in (3), § 38.

Invariance of the Connectivities R_i

42. We are now ready to prove the invariance of the connectivities $R_0, R_1, \cdots, R_n$ under the group of all homeomorphisms. This invariance is obvious for R_0 because R_0 is the number of connected complexes which compose C_n. To prove the invariance of $R_i (i>0)$ for any complex C_n, we first observe that according to § 23, R_i is the same for C_n as for any regular subdivision of C_n. We therefore fix attention on a regular subdivision $\bar{C}_n$.

By § 9 there exists* a set of i-circuits C_i^j $(j=1, 2, \cdots, R_i - 1)$ such that (1) there is no $(i+1)$-dimensional complex composed of cells of $\bar{C}_n$ which is bounded by any combination of the circuits C_i^j and (2) if C_i' is any other i-circuit composed of cells of $\bar{C}_n$ it is homologous to the sum (mod. 2) of some or all of the i-circuits C_i^j. By combining (1) with the theorem of § 39 we have at once that: (*a*) *there is no* $(i+1)$-*dimensional complex of any sort on* C_n *which is bounded by any combination of the circuits* C_i^j. From (2) and § 38 it follows that: (*b*) *if* C_i *is any i-circuit on* $\bar{C}_n$ *it is homologous to a linear combination (mod. 2) of the i-circuits* C_i^j $(j=1, 2, \cdots, R_i - 1)$. For C_i is homologous either to zero or to an i-circuit C_i' which is composed of cells of $\bar{C}_n$, and by (2) C_i' is homologous to a combination of the i-circuits C_i^j.

From the properties (*a*) and (*b*) it follows by a mere re-

---

* This is not intended to exclude the case in which $R_i - 1 = 0$, in which the set of i circuits C_i^j is a null-set.

petition of the argument in § 48, Chap. II that R_i *is an Analysis Situs invariant of the complex* C_n.

43. It should perhaps be pointed out explicitly that the proof which has just been completed applies as well for $i = n$ as for other values of i. If C_n is a single n-circuit, $R_n = 2$, and since R_n is an invariant, any complex C_n' homeomorphic with C_n contains just one n-circuit. By a repetition of the argument in § 52, Chap. II, it follows that this n-circuit contains all points of C_n'. *Hence any complex homeomorphic with an n-circuit is an n-circuit.*

CHAPTER IV

ORIENTABLE MANIFOLDS

Oriented n-Cells

1. Let us now take up the orientation of n-dimensional complexes. The first problem is to give a definition of the term "oriented n-cell." We shall give a definition here which suffices for the elementary part of the matrix theory and shall postpone to the next chapter the theorems on deformation which give the full intuitional content of the notion of orientation. The definition will be made as a part of a process of mathematical induction in which we prove that if certain theorems are true and certain terms defined for all complexes C_i for which $i < n$, then the theorems are true and the terms can be defined for any complex C_n. Since the theorems and definitions in question have already been established for all linear graphs, C_1, this process will establish them for all complexes C_n.

The terms which we assume to be defined are: oriented i-cell of a complex C_j $(i, j < n)$ orientable i-circuit $(i < n)$, oriented i-circuit $(i<n)$, oriented i-dimensional complex $(i<n)$, sum of oriented i-dimensional complexes $(i < n)$. The theorems are: (1) *any i-circuit* $(i < n)$ *which is homeomorphic with an orientable i-circuit is orientable*; (2) *any i-circuit defining an i-dimensional sphere* $(i < n)$ *is orientable.*

2. The proof that these theorems hold for any C_n if they hold for all C_i $(i < n)$ is a direct generalization of the proof given in §§ 58 to 60, Chap. II for the case $n = 2$, and will be given in § 10. Before establishing the theorems we state the definitions which, it will be noted, derive their content from the theorems for the cases $i < n$.

An *oriented n-cell* of a complex C_n is the object obtained by associating a cell a_i^n $(i = 1, 2, \cdots, \alpha_n)$ of C_n with one

of the oriented $(n-1)$-circuits which can be formed from its boundary according to the theorem (2) of the last section. One of the oriented n-cells formed from a_i^n is denoted by σ_i^n and the other by $-\sigma_i^n$. If σ_i^{n-1} is one of the oriented $(n-1)$-cells in the $(n-1)$-circuit associated with σ_j^n, σ_i^{n-1} is said to be *positively related* to σ_j^n and *negatively related* to $-\sigma_j^n$. If σ_i^{n-1} is an oriented $(n-1)$-cell of the $(n-1)$-circuit associated with $-\sigma_j^n$, it is said to be *negatively related* to σ_j^n and *positively related* to $-\sigma_j^n$.

The object obtained by assigning orientations to the n-cells of an n-dimensional complex is called an *oriented n-dimensional complex*. It is denoted by a symbol $(x_1, x_2, \cdots, x_{\alpha_n})$ in which $x_i = +1$ $(i = 1, 2, \cdots, \alpha_n)$ if σ_i^n is in the set, $x_i = -1$ if $-\sigma_i^n$ is in the set and $x_i = 0$ if neither of them is in the set. The *sum* of two such symbols is defined as in § 45, Chap. I; and if the sum of the symbols for two oriented complexes, Γ_n', Γ_n'' is the symbol for an oriented complex Γ_n''', the complex Γ_n''' is called a *sum** of Γ_n' and Γ_n'' and denoted by $\Gamma_n' + \Gamma_n''$.

Matrices of Orientation

3. Let orientations be assigned in arbitrary fashion to all the cells of a complex C_n. Then we can form an *incidence matrix*

$$E_k = \|\varepsilon_{ij}^k\| \quad (i = 1, 2, \cdots, \alpha_{k-1};\ j = 1, 2, \cdots, \alpha_k)$$

where ε_{ij}^k is $+1$, -1 or 0 according as $+\sigma_i^{k-1}$, $-\sigma_i^{k-1}$, or neither, is positively related to σ_j^k. We note that E_k is formed from H_k (§ 4, Chap. III) by changing some of the 1's into -1's. If the notation is changed so as to interchange the meanings of σ_j^k and $-\sigma_j^k$, the elements of the jth column of E_k and of the jth row of E_{k+1} must be multiplied by -1.

Covering Oriented Complexes

4. As in § 9, Chap. I and § 33, Chap. II, a generalized complex C_n' (which may be singular) on a complex C_n is said

* This term is given a more extensive significance in § 5 below.

to *cover* C_n if there is at least one point of C_n' on each point of C_n and there exists for every point of C_n' a neighborhood (§ 24, Chap. III) which is a non-singular complex on C_n. In case the number of points of C_n' which coincide with a given point of C_n is finite and equal to m for every point of C_n, C_n' is said to *cover* C_n *m times.*

In order to extend the notion of covering to oriented complexes we find it necessary to introduce the conception of cells coinciding both as sets of points and in orientation. Let C_k be a complex on C_n and such that each cell of C_k covers a cell of C_n just once. If the cells of both complexes are oriented, an oriented cell $\overline{\sigma}_p^i$ of C_k will be said to *coincide* with an oriented cell σ_q^i of C_n if and only if (1) $\overline{\sigma}_p^i$ is formed from an i-cell $\overline{a}_p^i$ which covers a_q^i and (2) the oriented $(i-1)$-cells formed from $(i-1)$-cells of C_k and positively related to $\overline{\sigma}_p^i$ coincide with oriented $(i-1)$-cells formed from $(i-1)$-cells of C_n and positively related to σ_q^i. This definition must be taken in the inductive sense. That is, since the meaning of "coincide" as applied to oriented 1-cells has already been established (§ 43, Chap. I), this statement when read with $i = 2$ defines it for oriented 2-cells, and so on.

5. The symbol $(x_1, x_2, \cdots. x_{\alpha_k})$ in which the x's are positive or negative integers will be used to denote a set of oriented k-cells in which there are $|x_i|$ $(i = 1, 2, \cdots, \alpha_k)$ oriented k-cells coincident with σ_i^k if x_i is positive or zero, and with $-\sigma_i^k$ if x_i is negative or zero.

The object obtained by assigning orientations to the k-cells of a singular k-dimensional complex is called a *singular oriented k-dimensional complex.** We shall generally omit the word "singular" when referring to it. The set of oriented k-cells of any singular oriented k-dimensional complex whose cells coincide with cells of the basic n-dimensional complex, determines a symbol $(x_1, x_2, \cdots, x_{\alpha_k})$. Conversely, any such

* Closely allied to this idea is that of a k-chain, a term introduced by J. W. Alexander and now in common use (1930). The k-chain is simply a linear combination, with integral coefficients, of symbols for oriented k-cells.

symbol determines at least one oriented k-dimensional complex. The latter can in general be constructed in a variety of ways, depending on how we join the set of k-cells represented by the symbol, by adding boundary cells of lower dimensions.

If two oriented complexes C_k and C'_k are formed from complexes whose cells have no common points unless they and their boundaries coincide, the two complexes can be regarded as on a third complex, and the two oriented complexes may be denoted by $(x_1, x_2, \cdots, x_{\alpha_k})$ and $(y_1, y_2, \cdots, y_{\alpha_k})$ respectively, the x's and y's being positive or negative integers or zero. An oriented (possibly singular) complex which can be denoted by $(x_1+y_1, x_2+y_2, \cdots, x_{\alpha_k}+y_{\alpha_k})$ is called a *sum* of C_k and C'_k and is denoted by $C_k+C'_k$.

In case the numbers $x_1, x_2, \cdots, x_{\alpha_k}$ have a common factor, so that

$$(x_1, x_2, \cdots, x_{\alpha_k}) = (pz_1, pz_2, \cdots, pz_{\alpha_k}),$$

any oriented complex whose symbol is $(z_1, z_2, \cdots, z_{\alpha_k})$ is said to be *covered p times* by a complex with symbol $(x_1, x_2, \cdots, x_{\alpha_k})$ formed by orienting the cells of a complex covering $(z_1, z_2, \cdots, z_{\alpha_k})$ p times in the sense of § 4.

Boundary of an Oriented Complex

6. By the *boundary* of an oriented i-cell σ^i_p is meant the oriented $(i-1)$-circuit associated with it in defining its orientation. Hence the pth column of the matrix E_i is the symbol $(y_1, y_2, \cdots, y_{\alpha_{i-1}})$ for the boundary of σ^i_p. The symbol for σ^i_p itself is $(x_1, x_2, \cdots, x_{\alpha_i})$ provided that $x_p = 1$ and $x_j = 0$ if $j \neq p$. Hence if

$$(1) \qquad E_i \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_i} \end{matrix} \right\| = \left\| \begin{matrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{\alpha_{i-1}} \end{matrix} \right\|,$$

where $(x_1, x_2, \cdots, x_{\alpha_i})$ is the symbol for σ^i_p, then $(y_1, y_2, \cdots, y_{\alpha_{i-1}})$ is the symbol for the boundary of σ^i_p.

By the *boundary* of any oriented i-dimensional complex we mean any sum of the boundaries of the oriented i-cells which compose it. If the boundary has symbol $(0, 0, \cdots, 0)$ we say that the given complex has no boundary.* From the identity,

$$(2) \qquad E_i \cdot \left\| \begin{matrix} x_1+x_1' \\ x_2+x_2' \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_i}+x_{\alpha_i}' \end{matrix} \right\| = E_i \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_i} \end{matrix} \right\| + E_i \cdot \left\| \begin{matrix} x_1' \\ x_2' \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_i}' \end{matrix} \right\|$$

it follows that if in the equation (1) $(x_1, x_2, \cdots, x_{\alpha_i})$ represents any i-dimensional complex, $(y_1, y_2, \cdots, y_{\alpha_{i-1}})$ represents its boundary. All this can be applied to the singular complexes discussed in the previous section.

7. In case the numbers $y_1, y_2, \cdots, y_{\alpha_{i-1}}$ in equation (1) have a common factor, so that

$$(y_1, y_2, \cdots, y_{\alpha_{i-1}}) = (kz_1, kz_2, \cdots, kz_{\alpha_{i-1}}),$$

the equation (1) signifies that the boundary of $(x_1, x_2, \cdots, x_{\alpha_i})$ may be taken as an oriented complex which covers the oriented complex denoted by $(z_1, z_2, \cdots, z_{\alpha_{i-1}})$ k times. An example of what this signifies geometrically may be constructed as follows:

Let S be the interior of a circle c in a Euclidean plane. Let F_n be a correspondence in which each point of c corresponds to the point obtained by rotating it about the center of c in a fixed sense through an angle of $2\pi/n$. The points of c are thereby arranged in sets of n such that each point of a set is carried by F_n into another point of the same set. All points in a set will be said to be "congruent." Let S_n be the set of objects consisting of the points of S and the sets of n points determined by F_n, each set of n congruent points being regarded as one object.

* The k-chain corresponding to any oriented complex without boundary is called a k-cycle under recent usage (1930). The term cycle was used by Poincaré.

The set of points S_n can be decomposed into a complex C_2 by the straight 1-cells joining the center of c to the $2n$ points of c in two of the sets of n congruent points. It is thus easily verified that the 2-cells of C_2 may be so oriented that their boundary is an oriented 1-circuit of $2n$ oriented 1-cells which covers an oriented circuit composed of two oriented 1-cells n times.

In case $n = 2$, the sets of n points are the diametrically opposite pairs of points of c, and S_n is homeomorphic with the projective plane.

Orientable n-Circuits

8. An n-circuit is said to be *orientable* or *non-orientable* according as the symbols for the boundaries of its oriented n-cells are, or are not, linearly dependent. If an orientable n-circuit has the property that each of its $(n-1)$-cells is incident with two and only two of its n-cells, orientations can be assigned to its n-cells in such a way that their sum has no boundary. This is proved as in § 56, Chap. II. The oriented complex formed in this way from such an orientable n-circuit, is called an *oriented n-circuit.*

If C_n is an orientable n-circuit, there is a singular oriented n-circuit K_n such that each cell of K_n is coincident with a cell of C_n, and each cell of C_n is coincident with at least one cell of K_n. To prove this we observe first that we can construct a singular oriented complex, without boundary, whose cells coincide with cells of C_n, as follows easily from the definition of orientable circuit. Next we prove, as in § 33, Chap. III, that the new oriented complex is a singular set of singular oriented generalized manifolds, each without boundary. The facts that orientation must be considered and that the initial complex is itself singular, cause no difficulty in this proof. Now let K_n be one of these oriented generalized manifolds, hence also an oriented n-circuit. The argument of § 56, Chap. II, shows that each cell of C_n must be coincident with at least one cell of K_n. This completes the proof.

By "singular oriented n-circuit" we shall usually mean one that is singular only in that a number of its cells may coincide with one cell of the basic complex. As with arbitrary oriented complexes, we shall usually omit the word "singular" when referring to it.

9. By reference to § 22, Chap. III, it is obvious that the boundaries of the n-cells into which an n-cell a_i^n of C_n is decomposed by a regular subdivision can be converted into a set of oriented $(n-1)$-circuits whose sum is the oriented $(n-1)$-circuit Γ_{n-1}^i formed from the boundary of a_i^n. Hence an n-circuit C_n is orientable if and only if a regular subdivision $\bar{C}_n$ of it is orientable.

It can now be proved by exactly the method used in §§ 58, 59, Chap. II that *if G_n is any n-circuit homeomorphic with C_n, G_n is orientable if and only if C_n is orientable.* In outline, the proof is as follows:

Let K_n be the n-circuit on C_n whose cells are the images under the homeomorphism of the cells of G_n. Let $\bar{C}_n$ be a regular subdivision of C_n and $\bar{K}_n$ a complex obtained from K_n by regular subdivisions as in § 36, Chap. III. Also let a correspondence A and a set of $(n+1)$-cells b_j^{n+1} $(j=1, 2, \cdots, \beta_n)$ be defined as in § 36. Each b_j^{n+1} is bounded by an n-circuit S_n^j of relatively simple construction, which is readily seen to be orientable.

If K_n is orientable there can be formed from the $(n-1)$-circuits bounding the n-cells of $\bar{K}_n$ a set of oriented $(n-1)$-circuits Γ_{n-1}^i $(i=1, 2, \cdots, \beta_{n-1})$ a linear combination of which sums to zero. From the $(n-1)$-circuits bounding the n-cells of each S_n^j is formed a set of oriented $(n-1)$-circuits whose sum is zero which contains one oriented $(n-1)$-circuit which is the negative of one of the Γ_{n-1}^i's. On adding multiples of all the oriented $(n-1)$-circuits thus obtained from $\bar{K}_n$ and the spheres S_n^j, all the oriented $(n-1)$-cells cancel out except some formed from the boundaries of the n-cells of $\bar{C}_n$. The latter are present because otherwise $\bar{K}_n$ would bound an $(n+1)$-dimensional complex on $\bar{K}_n$. These oriented $(n-1)$-circuits of $\bar{C}_n$ are subject to the linear relation

obtained by adding the linear relation among the Γ_{n-1}^i's and multiples of the ones obtained from the spheres S_n^j.

By an argument just like that given in § 56, Chap. II this linear relation involves all the $(n-1)$-circuits which can be formed from the boundaries of n-cells of $\bar{C}_n$, and hence $\bar{C}_n$ and C_n are orientable n-circuits. Therefore, if G_n is orientable, so is C_n. The relation between C_n and G_n being reciprocal, it follows at once that if C_n is orientable, so is G_n.

10. The complex used to define an n-dimensional sphere in § 5, Chap. III, is obviously orientable. By the last section, any complex homeomorphic with this one is orientable. Hence any n-circuit which defines an n-dimensional sphere is orientable. As a corollary, any n-circuit bounding an $(n+1)$-cell is orientable.

This completes the proof that the two Theorems (1) and (2) of § 1 are true for C_n if they are true for all C_i $(i<n)$ and thus establishes the cycle of theorems and definitions in § 1 for all values of n.

11. In case C_n is an n-circuit the rank of E_n determines whether C_n is orientable or not. This follows from the definition of orientability; and is proved in the manner of § 56, Chap. II. The proof shows that *C_n is orientable if the rank of E_n is $\alpha_n - 1$ and it is not orientable if the rank of E_n is α_n.*

In consequence of the Theorem (1) of § 1, if one complex defining a manifold M_n is orientable so are all complexes defining M_n. It is therefore justifiable to call a manifold orientable or not according as a particular complex into which it can be decomposed is orientable or not. Hence the criterion given in the last section will determine by a finite number of steps whether a manifold is orientable or not.

Thus for example, it is quite easy to write down a set of matrices defining a real projective space of n dimensions and prove the theorem that a real projective space is orientable if n is odd and not orientable if n is even. A proof of this theorem which makes use of combinatorial ideas but not of the matrix notation is given by Dénes König in the

Proceedings of the International Congress of Mathematicians at Cambridge in 1912, Vol. 2, p. 129.

Oriented k-Circuits

12. An oriented k-circuit (§ 8) has no boundary. Hence the symbol $(x_1, x_2, \cdots, x_{\alpha_k})$ for any oriented k-circuit satisfies the set of linear equations

$$(E_k) \qquad \sum_{j=1}^{\alpha_k} \epsilon_{ij}^k x_j = 0 \quad (i = 1, 2, \cdots, \alpha_{k-1})$$

which are equivalent to the matrix equation

$$(3) \qquad E_k \cdot \begin{Vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_k} \end{Vmatrix} = 0.$$

Conversely, any solution of these equations in integers represents a set of (possibly singular) oriented k-circuits on C_n. This is proved as in § 8.

We note that the boundary of any oriented complex is itself an oriented complex without boundary, hence satisfies relations of the above sort. This is proved as in § 28, Chap. II.

From the proof in § 8 it follows that *the boundary of any oriented k-dimensional complex may be taken as a set of oriented $(k-1)$-circuits.*

13. Since each column of E_{k+1} is the symbol $(x_1, x_2, \cdots, x_{\alpha_k})$ for an oriented k-circuit, it satisfies the condition (3). Hence

$$(4) \qquad E_k \cdot E_{k+1} = 0 \qquad (k = 0, 1, \cdots, n-1).$$

Normal Form of E_k

14. Let the rank of E_k $(k = 0, 1, \cdots, n)$ be denoted by r_k. By the theory of matrices whose elements are integers (cf. § 9 of Appendix II) there exist square matrices C_{k-1} and D_k with integer elements and determinants ± 1, of α_{k-1} and α_k rows respectively, such that

$$C_{k-1}^{-1} \cdot E_k \cdot D_k = E_k^* \tag{5}$$

where E_k^* is a matrix of α_{k-1} rows and α_k columns all the elements of which are zero except the first r_k elements of the main diagonal, which are the invariant factors of E_k. We shall denote the elements of the main diagonal of E_k^* by d_j^k, and understand that $d_j^k = 0$ if $j > r_k$.

We shall prove in § 18 below that the matrices C_{k-1} and D_k of the preceding paragraph can always be found with determinants $+1$.

Equation (5) is equivalent to

$$E_k \cdot D_k = C_{k-1} \cdot E_k^*, \tag{6}$$

and (6) may be regarded as a set of α_k equations of the form,

$$E_k \cdot \begin{Vmatrix} x_{1j} \\ x_{2j} \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_k j} \end{Vmatrix} = \begin{Vmatrix} d_j^k\, y_{1j} \\ d_j^k\, y_{2j} \\ \cdot \\ \cdot \\ \cdot \\ d_j^k \cdot y_{\alpha_{k-1} j} \end{Vmatrix}, \tag{7}$$

in which $x_{1j}, x_{2j}, \cdots, x_{\alpha_k j}$ are the elements of the jth column of D_k and $y_{1j}, y_{2j}, \cdots, y_{\alpha_{k-1} j}$ those of the jth column of C_{k-1}. By § 7, this means that the jth column of D_k represents an oriented complex the boundary of which covers an oriented complex represented by the jth column of C_{k-1} a number of times equal to the jth element of the main diagonal of E_k^*.

Since D_k has determinant ± 1, it follows that the symbol for any given oriented k-dimensional complex whose cells coincide with cells of C_n is expressible in unique manner as a linear combination of the columns of D_k. A similar remark can be made about C_k.

15. Any oriented k-dimensional complex without boundary is easily seen to be a set of oriented k-circuits, some or all of which may have in common certain cells of dimensionalities less than k. This does not exclude their having certain

k-cells coincident, but not identical. We shall refer to such a complex as a set of oriented k-circuits.

By the same sort of reasoning as in § 9, Chap. III, we infer that any set of oriented k-circuits is linearly dependent on the members of any set of independent sets of oriented k-circuits in number $\alpha_k - r_k$. If the latter set of sets of oriented k-circuits has the further property that an arbitrary set of oriented k-circuits is a linear combination, with integral coefficients, of its members, then it is called a *complete set* of sets of oriented k-circuits. No more than $\alpha_k - r_k$ sets of oriented k-circuits ean be independent. Similar remarks can be made about sets of oriented k-circuits no combination of which bounds; and we can define *complete set* of non-bounding sets of oriented k-circuits. It is one containing the least possible number of members such that an arbitrary set of oriented k-circuits is a linear combination with integral coefficients of its members, plus a set of oriented k-circuits which bounds or a multiple of which bounds. Such a complete set contains $\alpha_k - r_k - r_{k+1}$ members, as may be shown in a manner similar to that used in Chap. III. Similarly, we define a *complete set* of sets of oriented k-circuits each of which bounds or has a multiple which bounds as one containing the least possible number of members such that an arbitrary set of oriented k-circuits having the property just mentioned is a linear combination, with integral coefficients, of its members. From the preceding results it follows that the number of elements in the complete set just defined cannot be less than r_{k+1}. We shall show that it is exactly r_{k+1}.

Next we observe that in

$$E_{k+1} \cdot D_{k+1} = C_k \cdot E_{k+1}^*$$

each of the first r_{k+1} columns of C_k, say the jth column, represents an oriented k-circuit or set or oriented k-circuits covered a certain number, d_j^{k+1}, of times by the boundary of an oriented complex represented by the jth column of D_{k+1}. Now, since there exist $\alpha_k - r_k - r_{k+1}$ sets of oriented k-circuits no combination of which bounds, it follows that there

are at most r_{k+1} independent sets of oriented k-circuits each of which bounds or has a multiple that bounds. Hence the symbol for any set of oriented k-circuits which bounds or a multiple of which bounds is linearly dependent on these r_{k+1} columns of C_k. But since C_k has determinant ± 1, any such symbol is uniquely expressible as a linear combination, with integral coefficients, of its columns. Consequently these first r_{k+1} columns of C_k must further represent a complete set of sets of oriented k-circuits which bound or multiples of which bound.

Since the last $\alpha_k - r_k$ columns of E_k^* are composed entirely of zeros, and the determinant of D_k is ± 1, the last $\alpha_k - r_k$ columns of D_k represent a complete set of sets of oriented k-circuits. (Compare Chap. I, § 49.)

Now we shall establish the existence of a complete set of non-bounding sets of oriented k-circuits. Let u_i denote the ith column of C_k, and v_j the jth column of the block of the last $s = \alpha_k - r_k$ columns of D_k. Then we will have

$$u_i = \sum_{j=1}^{s} a_{ij}\, v_j, \qquad i = 1, \cdots, r = r_{k+1}.$$

Let C and D be square matrices with determinants ± 1 such that $C^{-1} A D$ is the normal form for A, the matrix of the a_{ij}'s. Let us set

$$w_k = \sum_{j=1}^{s} D_{kj}\, v_j, \qquad k = 1, \cdots, s,$$

$$x_t = \sum_{j=1}^{r} C_{tj}\, u_j, \qquad t = 1, \cdots, r,$$

where D_{kj} and C_{tj} are elements of D^{-1} and C^{-1} respectively. Then the elements of

$$C^{-1} A D W,$$

where W represents a column composed of the w's, are of the form $e_i w_i$. They are also seen to be identical with the x's, hence represent sets of oriented k-circuits which are bounding or have multiples that bound. Since the u's are

independent, the rank of A is r, so that there are r of these w_i's. The remaining w's are therefore $s-r=\alpha_k-r_k-r_{k+1}$ in number. Since all the w's represent a complete set of sets of oriented k-circuits, and the r of them mentioned above represent sets of oriented k-circuits which are bounding or have multiples that bound, it follows that these $s-r$ of the w's represent a complete set of non-bounding sets of oriented k-circuits.

16. Now suppose we replace D_k by a new D_k formed as follows. It has the same first r_k columns as the old D_k. The next $\alpha_k-r_k-r_{k+1}$ columns represent a complete set of non-bounding sets of oriented k-circuits, just proved to exist. The final r_{k+1} columns are the same as the first r_{k+1} columns of C_k. Since we merely replaced the last α_k-r_k columns by sets of oriented k-circuits, equation (6) continues to hold. Therefore the proposed changed will be proved to be permissible if we show that the new D_k has determinant ± 1.

The symbol for any oriented k-dimensional complex is a linear combination, with integral coefficients, of the first r_k columns of D_k plus a symbol for a set of oriented k-circuits, as follows from the structure of the original D_k. The symbol for any set of oriented k-circuits is a linear combination of the remaining columns of D_k, as follows directly from the definitions of the two blocks of columns in question. Thus the symbol for any oriented k-dimensional complex is a linear combination, with integral coefficients, of the columns of the new D_k. Consequently the determinant of the new D_k must be ± 1. We conclude that D_k can be modified as described above.

Thus we have determined D_k in such a way that: (1) each of its first r_k columns represents an oriented k-dimensional complex having a boundary which covers a set of oriented $(k-1)$-circuits a certain number d_i^k, of times; (2) each of the next $\alpha_k-r_k-r_{k+1}$ columns represents a set of oriented k-circuits which is not linearly dependent on sets of oriented k-circuits which bound, or multiples of which bound; (3) each

of the last r_{k+1} columns represents a set of oriented k-circuits covered a certain number, d_i^{k+1}, of times by the boundary of an oriented $(k+1)$-dimensional complex.

Since all but the first r_{k+1} rows of E^*_{k+1} consist of zeros the last $\alpha_k - r_{k+1}$ columns of C_k are arbitrary, subject to the condition that the determinant of C_k is to be ± 1. We arrange that C_k is obtainable from D_k by interchanging the blocks of columns (1) and (3).

The Betti Numbers

17. The number of oriented k-circuits in a complete set of non-bounding sets of oriented k-circuits is denoted by $P_k - 1$, so that

(8) $$P_k - 1 = \alpha_k - r_k - r_{k+1}.$$

The number P_k is called by Poincaré the kth *Betti number*.

It is readily verified that P_0 equals the number of connected parts of the complex. Hence P_0 equals R_0, and from § 39, Chap. I, it follows that

$$P_0 = \alpha_0 - r_1.$$

By the definition above,

$$P_k - 1 = \alpha_k - r_k - r_{k+1}$$

if $0 < k < n$, and

$$P_n - 1 = \alpha_n - r_n.$$

Multiplying these equations alternately by $+1$ and -1 and adding, we obtain

(9) $$P_0 + \sum_{k=1}^{n} (-1)^k (P_k - 1) = \sum_{k=0}^{n} (-1)^k \alpha_k$$

which may also be written

(10) $$\sum_{k=0}^{n} (-1)^k \alpha_k = \tfrac{1}{2}(-(-1)^n) + \sum_{k=0}^{n} (-1)^k P_k.$$

The expression on the left is the characteristic, and the formula is a generalization of Euler's formula. If C_n is connected, $P_0 = 1$. If further, C_n is an orientable n-circuit, as observed in § 10, $r_n = \alpha_n - 1$ and hence $P_n - 1 = 1$.

In this case, therefore,

$$
(11)\qquad \begin{aligned}
\sum_{k=0}^{n}(-1)^k\alpha_k &= 1+(-1)^n+\sum_{k=1}^{n-1}(-1)^k(P_k-1),\\
&= \tfrac{3}{2}(1+(-1)^n)+\sum_{k=1}^{n-1}(-1)^k P_k.
\end{aligned}
$$

Finally if C_n is a one-sided n-circuit, there is no solution of the equations (E_n) and hence $P_n-1=0$ and

$$
(12)\qquad \begin{aligned}
\sum_{k=0}^{n}(-1)^k\alpha_k &= 1+\sum_{k=1}^{n-1}(-1)^k(P_k-1)\\
&= \tfrac{1}{2}(3+(-1)^n)+\sum_{k=1}^{n-1}(-1)^k P_k.
\end{aligned}
$$

18. Now we shall prove that in the initial reduction to normal form (§ 14), the matrices C_{k-1} and D_k can both be taken with determinant $+1$. We know that every r_k, $0<k\leqq n$, is at least unity. It follows from (8), which holds for $0<k<n$, that for $0<k\leqq n$, r_k must be less than at least one of the numbers α_k and α_{k-1}. Consequently none of the matrices E_k can have its rank equal to the number of its columns and the number of its rows. Under these conditions the reduction to normal form can be done with matrices C_{k-1} and D_k whose determinants are $+1$, as is proved in § 9 of Appendix II.

However, we say nothing about the signs of the determinants of the C_k and D_k finally chosen, at the end of § 16.

19. Using the fact (§ 30, Chap. III) that for a manifold the characteristic is zero if n is odd, the equation (11) reduces to

$$
(13)\qquad P_1-P_2+\cdots-P_{n-1}=0 \qquad (n=2m+1)
$$

if C_n is an orientable manifold and (12) reduces to

$$
(14)\qquad P_1-P_2+\cdots-P_{n-1}=1 \qquad (n=2m+1)
$$

if C_n is a non-orientable manifold.

In the three-dimensional case (13) and (14) have the corollary that $P_1=P_2$ for an orientable manifold and

$P_1 = P_2 + 1$ for a non-orientable manifold. The first of these formulas is a special case of the duality formula obtained in § 39, below.

The Coefficients of Torsion

20. The numbers $d_1^k, d_2^k, \cdots, d_{r_k}$ defined in § 14 are such that the boundary of a complex represented by the ith column ($i = 1, 2, \cdots, r_k$) of D_k covers a $(k-1)$-circuit or set of $(k-1)$-circuits represented by the ith column of C_{k-1} d_i^k times.

Let us denote those of the numbers $d_1^k, d_2^k, \cdots, d_{r_k}^k$ which are not equal to 1 by

$$t_1^{k-1}, t_2^{k-1}, \cdots, t_{\tau_{k-1}}^{k-1}$$

the t's being arranged in such an order that each of them is the highest common factor of itself and all the t's which follow it. The numbers $t_1^{k-1}, t_2^{k-1}, \cdots, t_{\tau_{k-1}}^{k-1}$ are known as *the coefficients of torsion of dimensionality* $k-1$.

Each coefficient of torsion of dimensionality k is associated with a definite column of C_k which represents a set of oriented k-circuits such that the boundary of a $(k+1)$-dimensional complex covers them a number of times equal to the coefficient of torsion. Further details will be given in §§ 29 and 30 below.

It will be proved (§ 37) that the coefficients of torsion are topological invariants, and also that in case C_n defines an orientable manifold they satisfy a duality relation (§ 38).

21. It has been seen in § 50, Chap. I, that the invariant factors of the matrix E_1 are $+1$. Hence *there are no zero-dimensional coefficients of torsion.*

The matrix E_n in the case of an oriented manifold must have one $+1$ and one -1 in each row. But any such matrix can be regarded as the transverse of the matrix E_1 of a linear graph (§§ 17 and 38, Chap. I) and therefore has no invariant factors* except $+1$. Hence *an orientable manifold of n dimensions has no $(n-1)$-dimensional coefficients of torsion.*

* This theorem is also proved algebraically in Appendix II.

Relation between the Betti Numbers and the Connectivities

22. The matrices E_k reduce to the matrices H_k if all elements are reduced modulo 2. Hence if δ_k denote the number of even k-dimensional coefficients of torsion, the ranks of E_k and H_k are connected by the relation

$$r_k - \varrho_k = \delta_{k-1}.$$

Since

$$R_k - 1 = \alpha_k - \varrho_k - \varrho_{k+1}$$

and

$$P_k - 1 = \alpha_k - r_k - r_{k+1}$$

it follows that

(15) $$R_k - P_k = \delta_{k-1} + \delta_k$$

which is the formula for the connectivities in terms of the Betti numbers and the coefficients of torsion.

23. *The matrix E_n for a non-orientable circuit C_n has one coefficient of torsion, and this coefficient is even.* We prove this by means of (15), taking $k = n$. Since the left hand member is unity, and $\delta_n = 0$, we conclude that $\delta_{n-1} = 1$, that is, there is one even $(n-1)$-dimensional coefficient of torsion.

24. In the Monatshefte für Mathematik und Physik, Vol. 19 (1908), p. 49, a set of numbers, Q_k $(k = 0, 1, 2, \cdots, n)$, are defined by H. Tietze in terms which are superficially like our definition of the numbers R_k. But Tietze finds the formula (p. 56):

$$Q_k = P_k + \delta_{k-1}$$

which shows that the Q_k's as he used them are quite different from the R_k's.

Congruences and Homologies

25. The results obtained from the reduction of the matrices E_k to normal form will perhaps be clearer if they are restated in terms of another notation. Following Poincaré, we shall say that an oriented n-dimensional complex, Γ_n, is *congruent*

to a set of oriented $(n-1)$-circuits, Γ_{n-1}, if Γ_{n-1} is the boundary of Γ_n, and shall denote this relation by the symbols

$$(1) \qquad \Gamma_n \equiv \Gamma_{n-1}.$$

In case Γ_n has no boundary (i. e., is a set of n-circuits) Γ_n is said to be congruent to zero, and this is indicated by

$$(2) \qquad \Gamma_n \equiv 0.$$

The expressions (1) and (2) are called *congruences* and (2) is regarded as a special case of (1).

From § 6 it is evident that the sum of the left hand members of the two congruences is congruent to the sum of the right hand members. Moreover if both members of a congruence are multiplied by an integer, m, the resulting congruence,

$$(3) \qquad m\Gamma_n \equiv m\Gamma_{n-1}$$

has a meaning and is a consequence of (1) if we understand that $m\Gamma_n$ is an oriented complex which covers $\Gamma_n m$ times. If we understand that $-\Gamma_n$ stands for the oriented complex obtained from Γ_n by reversing the orientation of each of its cells, this statement can be extended to cover the cases in which m is negative. Hence *any congruence derived from a set of valid congruences of the same dimensionality by forming a linear homogeneous combination of them with integral coefficients is a valid congruence.*

26. Whenever the congruence

$$(4) \qquad \Gamma_k \equiv \Gamma_{k-1}$$

is satisfied by an oriented complex Γ_k on C_n, Γ_{k-1} is said to be *homologous to zero,*

$$(5) \qquad \Gamma_{k-1} \sim 0.$$

The relation

$$\Gamma_{n-1} - \Gamma'_{n-1} \sim 0$$

is also written

$$(6) \qquad \Gamma_{n-1} \sim \Gamma'_{n-1}$$

and expressed in words by saying that Γ_{n-1} is homologous to Γ'_{n-1}. Since a homology can always be reduced to a

congruence it follows that homologies can be combined linearly according to the rules that hold for the linear combination of congruences. Since the boundary of an oriented k-dimensional complex is a set of oriented $(k-1)$-circuits, the homology (5) implies the congruenee

$$(7)\qquad \Gamma_{k-1} \equiv 0.$$

It should be noted that these definitions do not permit the operation of dividing the terms of a homology by an integer which is a common factor of the coefficients. In other words,

$$(8)\qquad p\Gamma_{k-1} \sim 0$$

does not necessarily imply (5). Thus we are dealing with what Poincaré calls "homologies without division."

The Fundamental Congruences and Homologies

27. The relations between the k-cells and the $(k-1)$-cells given by the matrix E_k are equivalent to the system of congruences

$$[E'_k]\qquad \sigma_j^k = \sum_{i=1}^{\alpha_{k-1}} \varepsilon_{ij}^k \sigma_i^{k-1} \qquad (j = 1, 2, \cdots, \alpha_k).$$

The matrix of this system of congruences is obtained from E_k by interchanging rows and columns. The symbol $(x_1, x_2, \cdots, x_{\alpha_k})$ was used in § 5 to denote an oriented k-dimensional complex

$$x_1 \sigma_1^k + x_2 \sigma_2^k + \cdots + x_{\alpha_k} \sigma_{\alpha_k}^k .$$

Hence any matrix equation,

$$(9)\qquad E_k \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{\alpha_k} \end{matrix} \right\| = p \left\| \begin{matrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{\alpha_{k-1}} \end{matrix} \right\|$$

is equivalent to the congruence

$$(10)\quad x_1 \sigma_1^k + x_2 \sigma_2^k + \cdots + x_{\alpha_k} \sigma_{\alpha_k}^k \equiv p(y_1 \sigma_1^{k-1} + y_2 \sigma_2^{k-1} + \cdots + y_{\alpha_{k-1}} \sigma_{\alpha_{k-1}}^{k-1}).$$

28. The fundamental congruences $[E_k']$ give rise to the fundamental homologies

$$\{E_k'\} \qquad \sum_{i=1}^{\alpha_{k-1}} \varepsilon_{ij}^k \sigma_i^{k-1} \sim 0 \qquad (j = 1, 2, \cdots, \alpha_k)$$

and the matrix equation (9) corresponds to the homology

$$(11) \qquad p(y_1 \sigma_1^{k-1} + y_2 \sigma_2^{k-1} + \cdots + y_{\alpha_{k-1}} \sigma_{\alpha_{k-1}}^{k-1}) \sim 0.$$

29. The reduction of E_k to normal form, as interpreted in §§ 15 and 16, gives rise to the following congruences and homologies:

$$\begin{array}{lll}
(K_1) & \Gamma_k^i \equiv \Gamma_{k-1}^i & (i = 1, 2, \cdots, r_k - \tau_{k-1}), \\
(K_2) & \Gamma_k^i \equiv t_{i-r_k+\tau_{k-1}}^{k-1} \Gamma_{k-1}^i & (i = r_k - \tau_{k-1} + 1, \cdots, r_k), \\
(K_3) & \Gamma_k^i \equiv 0 & (i = r_k + 1, \cdots, r_k + P_k - 1), \\
(K_4) & \Gamma_k^i \sim 0 & (i = \alpha_k - r_{k+1} + 1, \cdots, \alpha_k - \tau_k), \\
(K_5) & t_{i-\alpha_k+\tau_k}^k \Gamma_k^i \sim 0 & (i = \alpha_k - \tau^k + 1, \cdots, \alpha_k),
\end{array}$$

in which Γ_k^i is an oriented k-dimensional complex represented by the ith column of D_k, Γ_{k-1}^i a set of oriented $(k-1)$-circuits represented by the ith column of C_{k-1}.

The congruences (K_1) correspond to the columns of D_k in the class (1) of § 16 for which the corresponding values of d_i^k are $+1$.

The congruences (K_2) correspond to the columns of D_k in the first block for which the values of d_i^k are different from 1. Thus t_i^{k-1} is the ith $(k-1)$-dimensional coefficient of torsion.

The congruences (K_3) correspond to the second block of columns of D_k enumerated in § 16. The sets of oriented k-circuits $\Gamma_k^i (i = r_k + 1, \cdots, r_k + P_k - 1)$ constitute a *complete set of non-bounding sets of oriented k-circuits.* They have the property that no linear combination of them coincides with the boundary of any oriented $(k-1)$-dimensional complex composed of oriented cells coincident with cells of C_n.

The sets of oriented k-circuits in the homologies (K_4) correspond to those of the last r_{k+1} columns of D_k which are

identical with the first $r_{k+1} - \tau_k$ columns of C_k. They therefore appear in the right-hand members of the congruences analogous to (K_1) which are determined by the matrix E_{k+1}.

The sets of oriented k-circuits in the homologies (K_5) correspond to the last τ_k columns of D_k and also to the set of columns of C_k which correspond to the coefficients of torsion t_i^k and thus appear in the right-hand members of the congruences analogous to (K_2) which are determined by the matrix E_{k+1}.

30. As we have seen in § 16, if Γ_k is any set of oriented k-circuits whose cells are coincident with cells of C_n, it is expressible in the form

$$\Gamma_k = \sum_{i=1}^{\alpha_k - r_k} a_i \Gamma_k^{r_k+i},$$

in which the coefficients a_i are integers and the $\Gamma_k^{r_k+i}$'s are the sets of oriented k-circuits which appear in the congruences and homologies (K_3), (K_4), (K_5). But by means of the homologies (K_4) and (K_5) this reduces to

$$(2) \qquad \Gamma_k \sim \sum_{i=1}^{P_k-1} a_i \Gamma_k^{r_k+i} + \sum_{i=1}^{\tau_k} b_i \Gamma_k^{\alpha_k-\tau_k+i}$$

in which the a_i's are integers and each b_i is a non-negative integer whose absolute value is less than the coefficient of torsion, t_i^k.

With a slight change of notation this result may be stated as follows: *There exists a set of $P_k - 1$ sets of oriented k-circuits $\Gamma_k^1, \Gamma_k^2, \cdots, \Gamma_k^{P_k-1}$ and a set of τ_k sets of oriented k-circuits, $\Gamma_k^{P_k}, \Gamma_k^{P_k+1}, \cdots, \Gamma_k^{P_k+\tau_k-1}$ such that if Γ_k is any set of oriented k-circuits whose cells are coincident with cells of C_n it satisfies a homology,*

$$(3) \qquad \Gamma_k \sim \sum_{i=1}^{P_k-1} a_i \Gamma_k^i + \sum_{i=1}^{\tau_k} b_i \Gamma_k^{P_k-1+i}$$

in which the a_i's and b_i's are integers and each b_i is non-negative and less than the coefficient of torsion t_i^k. The sets of oriented k-circuits $\Gamma_k^{P_k}, \Gamma_k^{P_k+1}, \cdots, \Gamma_k^{P_k+\tau_k-1}$ satisfy the τ_k homologies

$$(4) \qquad t_i^k \Gamma_k^{P_k-1+i} \sim 0 \qquad (i = 1, 2, \cdots, \tau_k).$$

We shall now prove that *the coefficients a_i in* (3) *are uniquely determined,* and that *the b_i are unique modulo the respective corresponding coefficients of torsion.*

Let

$$(3') \qquad \Gamma_k \sim \sum_{i=1}^{P_k-1} a_i' \Gamma_k^i + \sum_{i=1}^{\tau_k} b_i' \Gamma_k^{P_k-1+i}$$

be another homology like (3). From (3) and (3′) we obtain a homology which we write as a congruence, taking G_{k+1} as a bounded complex:

$$(5) \qquad G_{k+1} \equiv \sum_{i=1}^{P_k-1} (a_i - a_i') \Gamma_k^i + \sum_{i=1}^{\tau_k} (b_i - b_i') \Gamma_k^{P_k-1+i}.$$

It follows that

$$(6) \qquad t_{\tau_k}^k \sum_{i=1}^{P_k-1} (a_i - a_i') \Gamma_k^i \sim 0;$$

and since the Γ_k^i appearing in (6) are linearly independent with respect to homologies, we infer that $a_i = a_i'$, hence that the first sum in (5) is zero.

Now G_{k+1} is a unique (§ 14) linear combination of the Γ_{k+1}'s appearing in the relations similar to (K_1), $\cdots$, (K_5) but with k replaced by $k+1$. Hence the boundary of G_{k+1} is a linear combination of the left-hand members of (K_4) and (K_5) for the original k. Since the symbol for this boundary is uniquely determined, it follows from § 14 that the latter combination is also unique. Consequently the $(b_i - b_i')$ in (5) are multiples of the corresponding t_i^k, and the proof is complete.

As a corollary, $\Gamma_k \sim 0$ *if and only if the a_i in* (3) *are all zero and the b_i are all zero when reduced to their least absolute values modulo the corresponding t_i^k.*

Bounding k-Circuits

31. The results which have just been derived from the matrices apply only to complexes composed of cells coincident

with cells of C_n. But they can be proved to be valid for all complexes on C_n by an argument which is closely analogous to that used in Chap. III for the modulo 2 case. This argument centers about the following problem: *Given a set of oriented i-circuits Γ_i on C_n, does there exist on C_n an oriented $(i+1)$-dimensional complex of which Γ_i is the boundary?* It is of course understood that Γ_i may have any singularities compatible with its being on C_n.

32. This problem is solved by the means employed for the corresponding problem in Chap. III for complexes without orientation. The oriented i-circuits Γ_i are obtained by properly orienting the cells of a set of i-circuits K_i on C_n. For the i-circuits K_i we make the regular subdivisions and define the correspondence A and the complex B_{i+1} as in §§ 36, 37, Chap. III. The complex $\bar{K}_i$ which is obtained from K_i by regular subdivisions may by § 9 be converted into a set of oriented i-crcuits $\bar{\Gamma}_i$.

The complex B_{i+1} has one and only one $(i+1)$-cell b_j^{i+1} incident with each i-cell of $\bar{K}_i$ and the $(i+1)$-cells and i-cells b_j^{i+1} and b_k^i of B_{i+1} are subject to the same incidence relations as the i-cells and $(i-1)$-cells of $\bar{K}_i$. Hence if B_{i+1} is converted into an oriented complex Γ_{i+1} by orienting each $(i+1)$-cell of B_{i+1} so as to be positively related to the corresponding i-cell of $\bar{\Gamma}_i$, each of the i-cells b_j^i of B_{i+1} will be so oriented as to be positively and negatively related to equal numbers of oriented $(i+1)$-cells of Γ_{i+1}. (We recall that in general we are dealing with singular complexes). Hence none of the oriented i-cells formed from b_j^i will appear in the boundary of Γ_{i+1}. This boundary is the sum of the boundaries of the oriented $(i+1)$-cells of Γ_{i+1} and therefore consists either of $\bar{\Gamma}_i$ alone or of $\bar{\Gamma}_i$ and an oriented i-dimensional complex, which we shall call Γ_i', each oriented cell of which coincides with a cell of $\bar{C}_n$. Thus

$$\Gamma_{i+1} \equiv \bar{\Gamma}_i + \Gamma_i' \tag{1}$$

and hence

$$\bar{\Gamma}_i \sim \Gamma_i' \tag{2}$$

where Γ'_i is either 0 or such that each of its cells coincides with a cell of $\bar{C}_n$.

Now $\bar{\Gamma}_i$ is formed by orienting a regular subdivision of the complex K_i from which Γ_i is obtained by a process of orientation. Therefore if the orientation of $\bar{\Gamma}_i$ is chosen properly we will have (cf. § 38, Chap. III),

(3) $$\bar{\Gamma}_i \sim \Gamma_i$$
and hence
(4) $$\Gamma_i \sim \Gamma'_i.$$

33. From the homology (4), in case Γ'_i is not zero, it follows that if m is an integer different from zero

(5) $$m\Gamma_i \sim 0$$
if and only if
(6) $$m\Gamma'_i \sim 0.$$

The homology (6) means that there is an oriented complex $\varDelta_{i+1}$ on C_n whose boundary, $\varDelta_i$, covers Γ'_i m times. But under these conditions there is an oriented complex Γ_{i+1}, *composed of oriented cells which coincide with cells of* $\bar{C}_n$, such that its boundary covers Γ'_i m times. Except for certain considerations which do not arise in the modulo 2 case, and cause no difficulty here, this is proved as in § 40, Chap. III. We shall give no further details here.

34. We now have the solution of the problem of § 31. For a method has been given by which if Γ_i is any set of oriented i-circuits on a complex C_n and $\bar{C}_n$ is a regular subdivision of C_n, one can find a set of oriented i-circuits Γ'_i whose oriented cells coincide with cells of the same dimensionality of $\bar{C}_n$, such that $\Gamma'_i \sim \Gamma_i$ and, moreover, we have proved that if Γ'_i satisfies any homology $m\Gamma'_i \sim 0$ it satisfies a congruence $\Gamma_{i+1} \equiv m\Gamma'_i$ in which Γ_{i+1} is an oriented complex composed of oriented cells which coincide with cells of $\bar{C}_n$. From this it follows that if the sets of oriented k-circuits $\Gamma_k^1, \Gamma_k^2, \cdots, \Gamma_k^{P_k+\tau_k-1}$ are defined as in § 30 every set of oriented k-circuits Γ_k satisfies a homology like (3) of § 30.

Invariance of the Betti Numbers and Coefficients of Torsion

35. The theory of homologies can now be reduced to that of commutative groups as follows: Let us regard the totality of sets of oriented k-circuits homologous to a given set of oriented k-circuits as a single object, g, which we will refer to as a *group element.* Let us define the sum of two of these group elements, g_1 and g_2, as the totality of sets of oriented k-circuits each of which is the sum of a set of circuits of g_1 and a set of circuits of g_2. With respect to the operation of addition the group elements evidently constitute a group, which we shall call G_k. The identity element of the group is of course the totality of sets of oriented k-circuits which are homologous to zero; we denote the identity element by 0.

By § 30 there is one and only one group element containing each set of k-circuits

$$\sum_{i=1}^{P_k-1} a_i \Gamma_k^i + \sum_{i=1}^{\tau_k} b_i \Gamma_k^{P_k-1+i} \tag{7}$$

each a_i being an integer and each b_i being a positive or zero integer less than the coefficient of torsion t_i^k, and these group elements are all distinct.

The group G_k is defined in a unique manner and is therefore an Analysis Situs invariant.

36. If a group element g is such that for some positive integer x,

$$xg = 0,$$

g is said to be of finite order, and the smallest value of x for which this relation holds is the order of g. If no such integer x exists, g is said to be of infinite order.

The statement containing the formula (7) implies that G_k is "generated" by a set of elements (generators) $g^1, g^2, \cdots, g^{\tau_k+P_k-1}$ having the following properties: (1) the first τ_k of them are of finite orders and the order of each is a multiple

of the order of each preceding one; (2) the remaining ones are of infinite order; (3) no relation

$$\sum_{i=1}^{\tau_k} b_i g^i + \sum_{i=1}^{P_k-1} a_i g^{\tau_k+i} = 0$$

is satisfied unless all the a_i's are zero and each b_i is a multiple of the order of the corresponding g^i. The statement that the g's generate G_k means that every element of G_k is a linear combination, with integral coefficients, of the g's, and conversely.

That the Betti numbers and coefficients of torsion are Analysis Situs invariants is a consequence of the fact that the number $P_k - 1$ and the orders of the g's of finite orders are invariants of G_k, that is, are the same for all sets of generators satisfying the conditions (1), (2), (3). This theorem about groups may be put in the following form: In any two sets of generators satisfying the conditions (1), (2), (3), there is the same number of generators of orders greater than any integer r. We now prove the latter theorem.

37. If this were not so, there would be two finite sets of generators, $g^1, g^2, \cdots,$ and $p^1, p^2, \cdots,$ and a positive integer r such that if $g^{\varrho+1}, \cdots, g^{\varrho+\alpha}$ are the generators of orders greater than r in the first set and $p^{\sigma+1}, \cdots, p^{\sigma+\beta}$ the generators of orders greater than r in the second set, then β is greater than α. Since the g's are generators, we have

$$p^{\sigma+i} = x_1^i g^1 + \cdots + x_{\varrho+1}^i g^{\varrho+1} + \cdots + x_{\varrho+\alpha}^i g^{\varrho+\alpha} \tag{8}$$
$$(i = 1, 2, \cdots, \beta),$$

in which the coefficients x_j^i are integers and the coefficient of each g of finite order is less than this order but not negative.

Since α is less than β, we can find a set of integers $r_1, \cdots, r_\beta$ having H. C. F. unity, such that on multiplying each of the equations (8) by the corresponding r_i and adding them all together, the coefficient of each $g^{\varrho+i} (i = 1, 2, \cdots, \alpha)$ comes out zero. Thus we have

$$r_1 p^{\sigma+1} + \cdots + r_\beta p^{\sigma+\beta} = \sum_{i=1}^{\beta} r_i x_1^i g^1 + \cdots + \sum_{i=1}^{\beta} r_i x_\varrho^i g^\varrho.$$

Denoting the order of g^ϱ by λ, it follows that

$$\lambda r_1 p^{\sigma+1} + \cdots + \lambda r_\beta p^{\sigma+\beta} = 0. \tag{9}$$

(In case there is no g^ϱ, (9) still holds, with $\lambda = 1$).

By condition (3) on the p's, the coefficient of each p of infinite order in (9) must be zero, and the coefficient of each p of finite order must be a multiple of this order. Hence all the coefficients in (9) are zero, or else their H. C. F. is greater than r; whereas by the derivation of (9) the H. C. F. of the coefficients is λ which is at least unity and at most equal to r, a contradiction.

We have thus established the theorem about commutative groups, from which it follows that the Betti numbers and the coefficients of torsion are Analysis Situs invariants.

Duality of the Betti Numbers and Coefficients of Torsion

38. The duality relation,

$$R_{n-i} = R_i \qquad (i = 0, 1, \cdots, n-1)$$

was proved (§ 29, Chap. III) by showing that if $H_k\,(k = 1, \cdots, n)$ are the incidence matrices of C_n and $\bar{H}_k\,(k = 1, 2, \cdots, n)$ those of a complex C_n' dual to C_n, then

$$\bar{H}_{n-i} = H_{i+1}' \qquad (i = 0, 1. \cdots, n-\text{i}),$$

where H_{i+1}' is the matrix obtained by interchanging the rows and columns of H_{i+1}.

Now if C_n is an orientable manifold, orientations can be assigned to the dual cells in such a way that the matrices $E_k\,(k = 1, 2, \cdots, n)$ satisfy the corresponding relations, namely

$$\bar{E}_{n-i} = E_{i+1}'.$$

This property is a consequence of the orientability of the manifold, and can be proved by an extension of the argument used in §§ 25–28, Chap. III.

Since the invariant factors of E'_{k+1} are the same as those of E_{k+1} and $\bar{E}_{n-k} = E'_{k+1}$ it follows that the invariant factors of $\bar{E}_{n-k}$ and of E_{k+1} are the same. Hence *the* $(n-k-1)$-*dimensional coefficients of torsion of an orientable manifold are the same as the k-dimensional ones.* In other words

$$t_j^k = t_j^{n-k-1} \quad (k = 1, 2, \cdots, n-1;\ j = 1, 2, \cdots, \tau_k).$$

39. In view of the equation (§ 22),

$$R_k - P_k = \delta_{k-1} + \delta_k$$

and the equation $R_{n-k} = R_k$ it follows from this that the Betti numbers of an orientable manifold satisfy the condition

$$P_{n-k} = P_k.$$

It should be noted particularly that while the relation $R_{n-k} = R_k$ is satisfied by the connectivities of any manifold the relation $P_{n-k} = P_k$ is restricted to the orientable manifolds.

CHAPTER V

THE FUNDAMENTAL GROUP AND CERTAIN UNSOLVED PROBLEMS

Homotopic and Isotopic Deformations

1. Let K_i be a generalized complex on a generalized complex C_n. A set of transformations $F_x (0 \leqq x \leqq 1)$ is called a *one-parameter continuous family of transformations* if each F_x for each number $x (0 \leqq x \leqq 1)$ is a transformation of K_i and if, for $0 \leqq x \leqq 1$ and P any point of K_i, $F_x(P)$ is a continuous function of x and P. A (1–1)-continuous transformation F of K_i into a generalized complex K' on C_n is called a *deformation on C_n* if there exists a continuous family of transformations $F_x (0 \leqq x \leqq 1)$ such that F_0 is the identity, $F_1 = F$, and each F_x transforms K_i into a complex on C_n.

For example, C_n may be taken to be a 2-cell and K_i to be a single point. The points $F_x(P)$ then constitute a 1-cell with its ends, the 1-cell being singular or not according to the properties of F_x. As another example, K_i may be taken to be a 1-cell with its ends, the complexes into which K_i is transformed then are all 1-cells and constitute a 2-cell and its boundary.

Under the conditions described above, the complex K_i is said to be *deformed* into the complex K' and the complexes into which K_i is transformed by the functions F_x $(0 < x < 1)$ are called the *intermediate positions* of K_i.

It is an obvious consequence of the definitions made that if F_1 is a deformation on C_n which carries K_i to a generalized complex K' on C_n and F_2 a deformation on C_n of K' and if F_3 is the resultant of F_1 and F_2, then F_3 is a deformation on C_n.

2. Following the nomenclature introduced in the Dehn-Heegaard article on Analysis Situs in the Encyklopädie we shall distinguish between isotopic and homotopic deformations. A deformation is called an *isotopy* or an *isotopic deformation* if it satisfies the following condition: It is a deformation of a non-singular generalized complex K_i $(i = 0, 1, \cdots, n)$ through a set of intermediate positions which are non-singular and homeomorphic with K_i into a non-singular generalized complex K_i' which is homeomorphic with K_i. A non-singular generalized complex is said to be *isotopic* with any complex into which it can be carried by an isotopy.

The term *homotopy* will be used to designate a deformation in the general sense of § 1, and two generalized complexes C' and C'' will be said to be *homotopic* if one can be carried into the other by means of a homotopy.

For example, consider two pairs of distinct points $A\,B$ and $C\,D$ of an open curve. It is always possible to find a one-parameter continuous family of transformations carrying A and B into C and D respectively, but it is not always possible to find one in which all intermediate positions of AB are pairs of distinct points. In particular, it is not possible to interchange A and B by an isotopy.

Isotopy and Order Relations

3. We shall now state without proof a series of theorems which establish the relation between the order relations and the isotopic deformations. Let us agree that if a deformation F carries a 0-cell a^0 into a 0-cell $\bar{a}^0$ and if σ^0 and $\bar{\sigma}^0$ are the oriented 0-cells obtained by associating a^0 and $\bar{a}^0$ respectively with $+1$, then σ^0 is said to be carried by F into $\bar{\sigma}^0$, and $-\sigma^0$ to be carried by F into $-\bar{\sigma}^0$. This determines what is meant by the deformation of an oriented n-cell.

4. The following propositions hold for the non-singular 1-cells of either an open or a closed curve. Any two 1-cells are isotopic and any two oriented 1-cells are homotopic. But the oriented 1-cells fall into two classes such that two oriented

1-cells of the same class are isotopic, whereas two oriented 1-cells of different classes are not isotopic. Either of these classes may be called a *sense class*, or an *orientation* class or a *sense of description of the curve*. All oriented 1-cells of a sense-class are said to be *similarly sensed* or *oriented* and *to have the same sense*. All the oriented 1-cells of an oriented 1-circuit, as defined in § 35, Chap. I, are in the same sense-class. If two similarly oriented 1-cells of a curve are both positively related to the same oriented 0-cell, one of the 1-cells is contained in the other. If of two similarly oriented 1-cells one is positively and the other negatively related to the same oriented 0-cell, either of the 1-cells can be deformed by an isotopy which leaves the 0-cell invariant so as to have no points in common with the other.

5. Any transformation of a closed curve into itself which transforms a sense-class into itself is said to preserve sense; otherwise it alters sense. A (1–1) continuous transformation which preserves sense is an isotopic deformation. The isotopic deformations of a curve into itself form a self-conjugate subgroup of index two of the group of homeomorphisms of the curve into itself. A (1–1) continuous transformation which alters sense has exactly two invariant points.

6. At the beginning of Chap. IV, an oriented 2-cell of a complex C_2 was defined by associating a 2-cell with a particular oriented 1-circuit of its boundary, i. e., with a particular set of oriented 1-cells. This definition was sufficient for the combinatorial theory in which it was used but is manifestly not flexible enough to correspond fully to the intuitional idea of an element of surface with a sensed boundary. A definition which satisfies this requirement is the following:

An *oriented 2-cell* is a 2-cell associated with a sense-class of its boundary. If we recall that all the oriented 1-cells of an oriented 1-circuit belong to the same sense-class it is clear that the present definition can be substituted for the one used in Chap. IV without changing any of the theorems there obtained. From this point onward we shall use the term oriented 2-cell according to its new definition.

7. The fundamental theorems on deformation of oriented 2-cells are closely related to the theorem of § 60, Chap. II, on the invariance of orientableness. They may be stated, without proof, as follows: Any two oriented 2-cells on the same complex are homotopic. The non-singular oriented 2-cells on an orientable two-dimensional manifold fall into two classes called *sense-classes*, such that any two oriented 2-cells of the same sense-class are isotopic, and no two oriented 2-cells of different sense-classes are isotopic. Any oriented 2-cell and its negative belong to opposite sense-classes. Two isotopic oriented 2-cells are said to be *similarly oriented*. Two oriented 2-cells which have no point in common and are one positively and one negatively related to an oriented 1-cell are similarly oriented. Any two oriented 2-cells of a one-sided manifold are similarly oriented. Any one-sided manifold contains a Möbius strip.

The theorems in § 5 have been generalized to two dimensions by L. E. J. Brouwer, H. Tietze, and J. Nielsen, who have obtained a number of interesting theorems on the continuous transformations of two-dimensional manifolds and have also uncovered a number of interesting problems. The work of Brouwer and Nielsen can be found in recent volumes of the Mathematische Annalen, and further references to the literature can be found in an article on the subject of orientation by Tietze in the Jahresbericht der Deutschen Math. Ver., Vol. 29 (1920), p. 95.

8. It is obvious that the theorems in the first paragraph of § 7 form the basis for a generalization to n dimensions. An oriented 3-cell is defined as a 3-cell associated with a sense-class of its boundary. A set of theorems analogous to those just quoted for 2-cells hold for oriented 3-cells and form the basis of a definition of an oriented 4-cell, and so on.

9. In a regular complex an oriented n-cell may be denoted by the order in which its vertices $P_0, P_1, \cdots, P_n$, are written, with the convention that any even permutation of the vertices represents the same oriented n-cell and any odd permutation represents its negative. We have not had to use this

notation, and mention it only because it is useful in applications. Its significance may be said to depend on the following theorem:

The complex composed of an n-dimensional simplex (§ 1, Chap. III) and the k-dimensional simplexes determined by sets of $k+1$ ($k<n$) of its vertices can be isotopically deformed into itself in such a way that each of the k-dimensional simplexes goes into a k-dimensional simplex and the vertices are subjected to an arbitrary even permutation. This complex cannot be isotopically deformed into itself in such a way that each of its k-cells goes into one of its k-cells and so that the vertices are subjected to an odd permutation.

The Indicatrix

10. Another point of view from which the orientable manifolds may be considered is the following: Let A be a point of a manifold M_n and consider the set of all non-singular oriented n-cells on M_n which contain A. It can be proved that any such oriented n-cell can be deformed into any other such n-cell or into its negative through a set of intermediate positions which are all non-singular oriented n-cells containing A.* Moreover, no such oriented n-nell can be thus deformed into its negative. Each of the two classes of oriented n-cells thus determined for the point A is called an *indicatrix*, and the two indicatrices are called *negatives* of each other.

Now consider an isotopic deformation of a point A and its indicatrices. This carries A along a curve to a point A' and also a given indicatrix of A into an indicatrix of A'. If there is any closed curve along which A can be carried in such a way that one of the indicatrices at A is deformed into its negative, then M_n is one-sided. If not, M_n is orientable.

Another way of stating this result is as follows: Let a point associated with one of its indicatrices be called an indicatrix-point. In the case of an orientable manifold M_n the indicatrix-

* This has not yet been proved (1930).

points consist of two manifolds, each of which covers M_n once. In case M_n is one-sided the indicatrix-points constitute a single manifold which covers M_n twice.

A rather full discussion of the indicatrix, together with references to the literature is given by E. Steinitz, Sitzungsberichte der Berliner Mathematischen Gesellschaft, 7. Jahrgang (1908), p. 29. (Cf. footnote on page 71 above.)

11. These covering manifolds can also be obtained directly from the cellular structure of the complex C_n defining M_n. Let us form a complex C_n by the following rule: The i-cells $(i = n-1, n)$ of C_n are to be the $2\alpha_i$ oriented i-cells σ_j^i and $-\sigma_j^i$ $(j = 1, 2, \cdots, \alpha_i)$. For each oriented $(n-1)$-cell σ_j^{n-1}, there are four oriented n-cells, $\sigma_p^n, \sigma_q^n, -\sigma_p^n, -\sigma_q^n$, which are positively or negatively related to it. These fall into two pairs, one pair containing σ_p^n and the other containing $-\sigma_p^n$, such that one of the oriented n-cells of a pair is positively related while the other is negatively related to σ_j^{n-1}. Let both oriented n-cells of one pair be incident with σ_j^{n-1} and let both the oriented n-cells of the other pair be incident with $-\sigma_j^{n-1}$ in the set of incidence relations defining C_n. A singular n-circuit or pair of n-circuits which is thus defined on M_n can easily be proved to be either one or two manifolds, each of which covers M_n.

If there are two of these manifolds, M_n is orientable; and if there is only one, M_n is one-sided or non-orientable. Moreover, each oriented $(n-1)$-cell of C_n is positively related to one oriented n-cell of C_n and negatively related to one other. Hence the covering manifold or manifolds are orientable.

12. The covering manifolds which are referred to above must be distinguished clearly from Riemann surfaces. The latter are surfaces on a sphere which have the properties of covering surfaces except at a finite number of points, the branch points. It would be easy to develop the topological part of the theory of Riemann surfaces at this point by the methods which we have been using, and this would doubtless be done in a more extensive treatise.

It is well known that any orientable two-dimensional manifold can be regarded as a Riemann surface. The definition of a Riemann surface has been generalized to n dimensions by P. Heegard and it has been proved by J. W. Alexander (Bull. Amer. Math. Soc., Vol. 26 (1920), p. 370) that any orientable manifold can be regarded as a Riemann manifold.

Theorems on Homotopy

13. By a slight modification of the argument in §§ 35 to 41, Chap. III, it can be proved that *any k-dimensional complex C_k on a regular n-dimensional complex $\overline{C}_n$ is homotopic with a complex C_k' consisting of cells each of which covers* (§ 4, Chap. IV) *a cell of $\overline{C}_n$*. The nature of the modification needed will be sufficiently indicated by a consideration of the case of a 1-circuit K_1 on a regular complex $\overline{C}_n$. Let the definition of the 1-cell b_i^1 in § 37, Chap. III, be modified so that b_i^1 stands in each case for a 1-cell joining a vertex of $\overline{K}_1$ not to a vertex of $\overline{C}_n$ but to a point coincident with a vertex of $\overline{C}_n$. Likewise let the boundary of each b_i^2 be the same as in Chap. III except that if it contains a cell of $\overline{C}_n$ this cell is replaced by one coincident with it. Thus when the boundaries of the 2-cells b_i^2 are added (mod. 2) the only 1-cells cancelled are the 1-cells b_i^1. Hence the boundary of B_2 is the 1-circuit $\overline{K}_1$ and a 1-circuit or set of 1-circuits K_1' composed of 0-cells and 1-cells each covering a cell of C_n.

It is obvious that K_1 is homotopic with K_1'. For a definition of straightness and distance on B_2 can be made in such a way that each 2-cell of B_2 is a square with one side on K_1 and one on K_1' or a triangle with one side on K_1 and the opposite vertex on K_1'. Each point X of K_1 may then be joined to a point of K_1' by a straight 1-cell x in such a way that every interior point of B_2 is on one and only one of these 1-cells. A transformation F_t may be defined as that transformation which carries each point X of K_1 to the point P of the 1-cell x whose distance along x from X is to the length of the 1-cell x in the ratio t. The trans-

formations F_t evidently give a one-parameter continuous family which define a deformation of K_1 into K_1'.

14. A fundamental theorem of homotopy is the following: *If K_n is a non-singular n-circuit on an n-circuit C_n, then K_n cannot be deformed into a single point on C_n.* For if such a deformation of K_n were possible K_n would bound a singular $(n+1)$-dimensional complex on C_n composed of K_n, the point into which K_n was deformed, and all intermediate positions of K_n. This would be contrary to the first theorem in the second paragraph of § 41, Chap. III.

The Fundamental Group

15. The theory of the homotopy of curves on a complex leads to the important concept of the fundamental group. Let O be an arbitrary point of a complex C_n and consider all oriented 1-cells on C_n whose initial and final points coincide with O. These 1-cells may be singular in any way whatever.

Two of these oriented 1-cells which are such that one of them can be deformed into the other through a set of intermediate positions, all of which are oriented 1-cells of the set, are said to be *equivalent*. Let us denote oriented 1-cells of the set by g's with subscripts, as g_1, g_2, g_i, g_x, etc., with the convention that any two equivalent g's may be denoted by the same symbol. Hence by the usual convention on the equality sign, $g_1 = g_2$ means that any oriented 1-cell denoted by g_1 is equivalent to any one denoted by g_2. Also let any 1-cell which is the negative of one denoted by g_1 be denoted by g_1^{-1}. Finally let any 1-cell of the set which may be deformed into the point O through a set of intermediate positions which are all g's be denoted by 1. Thus the equation

$$g_x = 1$$

means that g_x may be deformed into coincidence with O through a set of g's.

16. If the terminal point of an oriented 1-cell g_1 is identical with the initial point of an oriented 1-cell g_2, the oriented

1-cell g_3 containing all points of g_1 and g_2 and having the same initial point as g_1 and terminal point as g_2 is denoted by $g_1 \cdot g_2$, and g_3 is called the *product* of g_1 and g_2.

This definition holds whether the initial and terminal points of g_1 coincide with the same point of C_n or not. For any g_1 and g_2 whose terminal and initial points respectively coincide with the same point of C_n there exists a g_3 such that

$$g_3 = g_1 \cdot g_2$$

because there always exists one of the oriented 1-cells denoted by g_2 which has the terminal point of g_1 as its initial point. It is also clear that, in general,

$$g_1 \cdot g_2 \neq g_2 \cdot g_1,$$

whereas

$$g_1 \cdot g_1^{-1} = 1,$$

and

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3.$$

17. The symbols g defined in § 15 can be regarded as the operations of a group. For there is a single-valued definition of multiplication, they satisfy the associative law, they include among themselves an identity operation, and there is a unique inverse in the set for each operation. This group is known as the *fundamental group* of the complex C_n.

The fundamental group of any complex C_n is independent of the choice of the point O. For let O' be any other point of C_n and let g_1 be an oriented 1-cell whose initial point coincides with O' and whose terminal point coincides with O. If g_x is any oriented 1-cell whose initial and terminal points coincide with O,

$$\bar{g}_x = g_1 \cdot g_x \cdot g_1^{-1}$$

is one whose initial and terminal points coincide with O'. A similar remark can be made with the roles of O and O' interchanged, and it is seen that in this way a one-to-one correspondence is set up between the group elements g_x defined with respect to the point O and the group elements $\bar{g}_x$ defined

with respect to the point O'. The operations $\bar{g}_x$ form a group which is isomorphic with that formed by the operations g_x because

$$\begin{aligned} \bar{g}_x \cdot \bar{g}_y &= g_1 \cdot g_x \cdot g_1^{-1} \cdot g_1 \cdot g_y \cdot g_1^{-1} \\ &= g_1 \cdot g_x \cdot g_y \cdot g_1^{-1}. \end{aligned}$$

The Group of a Linear Graph

18. The groups of two particular linear graphs should be noticed:

(1) In case C_1 is an open curve its group contains only one operation, the identity. For every closed curve on C_1 can be deformed into coincidence with O.

(2) In case C_1 is a simple closed curve its group contains an operation g which represents an oriented 1-cell which has just one point coincident with each point of C_1 except O. The complete set of operations may be represented by

$$\begin{array}{llllll} 1, & g, & g^2, & \cdots, & g^n, & \cdots \\ & g^{-1}, & g^{-2}, & \cdots, & g^{-n}, & \cdots \end{array}$$

where

$$g^2 = g \cdot g, \quad g^3 = g^2 \cdot g, \quad \text{etc.}$$

and

$$g^{-n} = (g^{-1})^n.$$

The operations are all distinct, as is easily proved by consideration of the total algebraic change of a coordinate measured along the curve.

19. In case C_1 is an arbitratry linear graph and a_i^1 a 1-cell joining two distinct 0-cells of C_1, the operation of *shrinking a_i^1 to a point* will be taken to mean the *operation of replacing* C_1 by a complex C_1' which is identical with C_1 except that a_i^1 and its ends have been replaced by a single 0-cell which is incident with every 1-cell with which either of the ends of a_i^1 was incident.

The operation of shrinking a_i^1 to a point does not change the characteristic of C_1. For it decreases α_1 and α_0 each by 1 and hence leaves $\alpha_0 - \alpha_1$ invariant. The operation

may make the boundaries of certain 1-cells singular by bringing their ends into coincidence. But by introducing a new 0-cell in the interior of each such 1-cell, the graph may be restored to a form in which each 1-cell has distinct ends. The operation of shrinking a 1-cell to a point obviously leaves R_0 invariant. Hence, by the formula,

$$\alpha_0 - \alpha_1 = R_0 - R_1$$

it leaves R_1 invariant. This is obvious also because the operation neither produces nor destroys 1-circuits.

The operation may be repeated so long as there are two distinct 0-cells joined by a 1-cell. In case C_1 has no 1-circuits, i. e., in case it consists of R_0 trees, the operation reduces C_1 to a set of R_0 0-cells.

In case C is not a tree and $R_0 = 1$, the result of repeating the operation of shrinking to a point $\alpha_0 - 1$ times is a linear graph consisting of R_1 closed curves having a single point in common. In case $R_0 > 1$ the result is R_0 such graphs.

20. It can easily be proved that the operation of shrinking a 1-cell a_i^1 of a linear graph C_1 to a point changes C_1 into a complex C_1' having the same group as C_1. (Compare § 18.)

From this it follows that: (1) The group of any tree is the identity; (2) the group of any complex which is not a tree is the same as the group of a complex consisting of $R_1 - 1$ 1-cells each having a 0-cell O as its initial and terminal point, and no two having a point in common. This group consists of $R_1 - 1$ operations g_i $(i = 1, 2, \cdots, R_1 - 1)$ and all combinations of them. Thus the general expression for an operation of the group is

$$g_1^{a_1} \cdot g_2^{a_2} \cdots g_\mu^{a_\mu} \cdot g_1^{b_1} \cdot g_2^{b_2} \cdots g_\mu^{b_\mu} \cdots g_1^{j_1} \cdot g_2^{j_2} \cdots g_\mu^{j_\mu}$$

where the exponents can be any integers, positive, negative or zero, and $\mu = R_1 - 1$.

21. The operations $g_1, g_2, \cdots, g_\mu$ are called the *generators* of the group of C_1. They are absolutely independent of each other, that is to say they satisfy no identities except the laws of combination given in § 16.

In the general theory of discrete groups having a finite number of generators the generators are supposed to satisfy certain identities of the form,

$$g_{i_1}^{m_1}\, g_{i_2}^{m_2} \cdots g_{i_k}^{m_k} = 1\,,$$

which are known as *generating relations.* The groups of n-dimensional complexes ($n \geqq 2$) will be seen usually to have generating relations. The group of a linear graph is thus characterized by the lack of generating relations.

The Group of a Two-dimensional Complex

22. Let C_2 be a two-dimensional complex, $\overline{C}_2$ a regular subdivision of it, and let C_1 be the linear graph composed of the 1-cells and 0-cells on the boundaries of the 2-cells of $\overline{C}_2$. Also let the point O which is the common end point of the generators g, of the fundamental group be a vertex of $\overline{C}_2$. By § 13 any 1-circuit on C_2 may be deformed into (is homotopic with) one composed of 0-cells and non-singular 1-cells coincident with 0-cells and 1-cells of C_1. *Hence the generators of the fundamental group of C_2 may be taken to be a set of generators of the fundamental group of C_1.*

Every 2-cell a_i^2 of $\overline{C}_2$ determines a relation among the g's. For let a be an oriented 1-cell whose initial point is O and whose terminal point, P, is on the boundary of a_i^2 and let b be an oriented 1-cell having P as initial and terminal points and coinciding in a non-singular way with the boundary of a_i^2. Then $a \cdot b \cdot a^{-1}$ is one of the g's and is expressible in terms of the generating operation of the fundamental group of C_1. Hence

$$a \cdot b \cdot a^{-1} = 1$$

is a relation among the generators of the fundamental group of C_2.

If a_k^2 is another 2-cell of C_2 whose boundary has an oriented 1-cell m_1 in common with the boundary of a_i^2, the boundaries of a_k^2 and a_i^2 can be expressed in the forms .

$$m_2 \cdot m_1 \quad \text{and} \quad m_1^{-1} \cdot m_3$$

respectively, where m_2 and m_3 are oriented 1-cells. The boundary of the 2-cell b^2 composed of a_k^2 and a_i^2 and the points of m_1 exclusive of its ends is then

$$m_2 \cdot m_1 \cdot m_1^{-1} \cdot m_3 = m_2 \cdot m_3.$$

Moreover a may be taken to be a 1-cell joining O to an end of m_1. The relation determined among the generators of C_2 by a_k^2 is therefore

(1) $$a \cdot m_2 \cdot m_1 \cdot a^{-1} = 1,$$

that determined by a_i^2 is

(2) $$a \cdot m_1^{-1} \cdot m_3 \cdot a^{-1} = 1$$

and that determined by b^2 is

(3) $$a \cdot m_2 \cdot m_1 \cdot a^{-1} \cdot a \cdot m_1^{-1} \cdot m_3 \cdot a^{-1} = a \cdot m_2 \cdot m_3 \cdot a^{-1} = 1.$$

The equation (3) is obviously a consequence of (1) and (2). Hence any 2-cell on $\bar{C}_2$ which is composed of cells coincident with the cells a_i^0, a_i^1, a_i^2 gives rise to a relation among the generators of the group which is a consequence of those determined by the 2-cells $a_1^2, a_2^2, \cdots, a_{\alpha_2}^2$.

But any 2-cell on C_2 is homotopic with one which is composed of cells coincident with 0-cells and 1-cells and 2-cells of $\bar{C}_2$. Hence any relation among the generators of the group is expressible in terms of the relations determined by the 2-cells of $\bar{C}_2$. Hence the group has α_2 generating relations, some of which, in general, are redundant.

23. In case C_2 defines a closed manifold M_2, its group G can be obtained in a simple form by considering C_2 reduced as in § 62, Chap. II, to a single 2-cell bounded by a linear graph C_1 in which there are $R_1 - 1$ linearly independent circuits. It follows readily that G is generated by $R_1 - 1$ generators connected by one generating relation. If C_1 is

further normalized as outlined in § 65 this relation may be reduced to one of the following three forms

$$(1) \quad a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdots a_p \cdot b_p \cdot a_p^{-1} \cdot b_p^{-1} = 1$$

$$(2) \quad a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdots a_p \cdot b_p \cdot a_p^{-1} \cdot b_p^{-1} \cdot c_1 \cdot c_1 = 1$$

$$(3) \quad a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdots a_p \cdot b_p \cdot a_p^{-1} \cdot b_p^{-1} \cdot c_1 \cdot c_1 \cdot c_2 \cdot c_2 = 1$$

in which the a's, b's and c's are generating operations and the relation (1) corresponds to a two-sided manifold of genus p, (2) to a one-sided manifold of the first kind, and (3) to a one-sided manifold of the second kind. The generating relations (2) and (3) can also be written in the from

$$(4) \qquad c_1^2 \cdot c_2^2 \cdots c_{R_1-1}^2 = 1$$

which is equivalent to (2) if $R_1 - 1 = 2p+1$ and to (3) if $R_1 - 1 = 2p+2$.

The fundamental group of a closed manifold is infinite except in the case of the sphere. for which the group is the identity, and of the projective plane, for which it consists of one operation of period two and the identity.

24. An important though obvious consequence of the last sections is that any discrete group with a finite number of generators is the fundamental group of a two-dimensional complex. For, given a group with n generators $g_1, g_2, \cdots, g_n$ and k generating relations, construct a linear graph C_1 consisting of a point O and n closed curves having O and no other points in common. Let one of these curves correspond to each generator. The left hand member of each generating relation denotes a closed curve on C_1. Introduce a 2-cell (whose boundary is in general singular) bounded by each of these curves. The result is a two-dimensional complex having the given group as its fundamental group.

The Commutative Group $\bar{G}$

25. Suppose that a group G is determined by n generators $g_1, g_2, \cdots, g_n$ and a number k of generating relations. The latter may be written in the form

$$
(1)\quad
\begin{array}{l}
g_1^{a_{11}}\cdot g_2^{a_{12}}\cdots g_n^{a_{1n}}\cdot g_1^{b_{11}}\cdot g_2^{b_{12}}\cdots g_n^{b_{1n}}\cdots g_1^{j_{11}}\cdot g_2^{j_{12}}\cdots g_n^{j_{1n}}=1,\\
g_1^{a_{21}}\cdot g_2^{a_{22}}\cdots g_n^{a_{2n}}\cdot g_1^{b_{21}}\cdot g_2^{b_{22}}\cdots g_n^{b_{2n}}\cdots g_1^{j_{21}}\cdot g_2^{j_{22}}\cdots g_n^{j_{2n}}=1,\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
g_1^{a_{k1}}\cdot g_2^{a_{k2}}\cdots g_n^{a_{kn}}\cdot g_1^{b_{k1}}\cdot g_2^{b_{k2}}\cdots g_n^{b_{kn}}\cdots g_1^{j_{k1}}\cdot g_2^{j_{k2}}\cdots g_n^{j_{kn}}=1.
\end{array}
$$

The exponents of the g's are positive or negative integers or zero. The group is characterized by the matrix of the exponents. This matrix has k rows and a number of columns which is a multiple of n. It will be called the *matrix of the group.*

If the group G is commutative, that is, if $g_i \cdot g_j = g_j \cdot g_i$ for all values of i and j, the left member of each expression in (1) can be written in the form

$$g_1^{a_{r1}} \cdot g_2^{a_{r2}} \cdots g_n^{a_{rn}}.$$

Hence in this case the matrix is one of k rows and n columns.

If G is not commutative there is a unique commutative group $\bar{G}$ associated with it, namely the group generated by $g_1, g_2, \cdots, g_n$ subject to the conditions (1) and the condition that all the operations are commutative. The matrix of $\bar{G}$ is

$$\|\gamma_{rs}\| \quad (r=1,2,\cdots,k;\ s=1,2,\cdots,n)$$

where

$$\gamma_{rs} = a_{rs} + b_{rs} + \cdots + j_{rs}.$$

26. Regarding G as the group of a two-dimensional complex, the commutative group $\bar{G}$ can be studied by means of the matrix E_2. For let the oriented 1-cells σ_i^1 $(i=1,2,\cdots,\alpha_1)$ be denoted by σ_i, and also, in the present section, denote the number α_1 by λ. Then each of the generators g_1, g_2, $\cdots$, g_n can be expressed in the form

$$(2)\quad g_i = \sigma_1^{\alpha_{i1}} \sigma_2^{\alpha_{i2}} \cdots \sigma_\lambda^{\alpha_{i\lambda}} \cdots \sigma_1^{\zeta_{i1}} \sigma_2^{\zeta_{i2}} \cdots \sigma_\lambda^{\zeta_{i\lambda}} \quad (i=1,2,\cdots,n).$$

On substituting these expressions in (1) we find the generating relations of G expressed in terms of the σ's. If the group

is set up in the manner described in § 22 each of these relations takes the form

(3) $$l_j m_j l_j^{-1} = 1$$

where l_j is a set of σ's representing a curve from O to a point on the boundary of one of the 2-cells a_j^2 and m_j represents the boundary of the 2-cell.

On passing to the group $\tilde{G}$ by introducing the condition of commutativity (3) becomes

(4) $$m_j = 1.$$

Since m_j represents the boundary of the 2-cell a_j^2 (4) is expressible in the form

(5) $$\sigma_1^{e_{1j}} \sigma_2^{e_{2j}} \cdots \sigma_\lambda^{e_{\lambda j}} = 1 \qquad (j = 1, 2, \cdots, \alpha_2)$$

in which the exponents are the elements of the jth column of the matrix E_2. Hence *the generating relations of the group* $\tilde{G}$ *when expressed in terms of* $\sigma_1, \sigma_2, \cdots, \sigma_{\alpha_1}$, *take the form* (5) *in which the matrix of the exponents is* E_2', *the matrix obtained by interchanging the rows and columns of* E_2.

It is worthy of comment that whereas the group G is defined in terms of a definite point O of C_2 (an isomorphic group is obtained from any other point O'), the group $\tilde{G}$ has no reference to any particular point O. This is because the terms l_j and l_j^{-1} in (3) cancel out when the assumption of commutativity is introduced.

27. The fundamental group G is such that

$$g_x = 1$$

signifies that the closed curve represented by g_x bounds a 2-cell on C_2. The geometric significance of the group $\tilde{G}$ is equally simple. If g_y is an element of this group

(6) $$g_y = 1$$

signifies that the closed curve or set of closed curves represented by g_y *bounds a two-dimensional complex on* C, or in other words,

(7) $$g_y \sim 0$$

where we now let g_y stand for the oriented curve obtained by identifying the initial and terminal points of the oriented 1-cell g_y. That (6) and (7) have the same geometrical significance is immediately evident if one compares the steps by which (6) is obtained from (5) with those by which (7) is obtained from the fundamental homologies of § 28, Chap. IV.

Equivalences and Homologies

28. The operation of combining two elements of a group which is called multiplication in the sections above can equally well be denoted by the sign $+$ and called addition. This is done in fact by Poincaré in a number of places. He thereby replaces any relation of the type (1) by

(8) $a_{i1}g_1 + a_{i2}g_2 + \cdots + a_{in}g_n + \cdots + j_{i1}g_1 + j_{i2}g_2 \cdots + j_{in}g_n \equiv 0$

which he calls an *equivalence*. In an equivalence the operation of addition is non-commutative. The equivalence (8) signifies that the elements on the left-hand member constitute the boundary of a 2-cell on C_2. To develop the theory of equivalence further would amount merely to repeating the theory of the group G in a different form. Any equivalence can be derived from the corresponding group identity by the formal process of taking logarithms.

Poincaré also makes use of a second class of equivalences which he calls *improper equivalences.* These are obtained from the proper equivalences by dropping the restriction that each 1-cell shall begin and end at O and allowing *cyclic permutation* of the terms of an equivalence. Thus if two 1-cells are properly equivalent they are homotopic by a deformation through intermediate positions each of which is a 1-cell whose ends coincide with O. If they are improperly equivalent they are homotopic in the general sense.

It should be noted that the equivalences and congruences (§ 25, Chap. IV) of Poincaré are entirely different notions although they are designated by the same notation.

29. Any equivalence (8) gives rise to a homology

(9) $$\gamma_{i1}g_1 + \gamma_{i2}g_2 + \cdots + \gamma_{in}g_n \sim 0$$

which is distinguished from (8) by the fact that the commutative law of addition holds good and by the fact that

$$\gamma_{ik} = a_{ik} + b_{ik} + \cdots + j_{ik} \qquad (k = 1, 2, \cdots, n).$$

The homologies thus correspond to the identities of the group $\tilde{G}$, which may well be called the homology group.

The Poincaré Numbers of G

30. Let us consider a set of n operations of G, $g'_1, g'_2, \cdots, g'_n$, where

$$(10) \qquad \begin{array}{l} g'_1 = g_1^{\alpha_{11}} \cdot g_2^{\alpha_{12}} \cdots g_n^{\alpha_{1n}} \cdots g_1^{\zeta_{11}} \cdot g_2^{\zeta_{12}} \cdots g_n^{\zeta_{1n}} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g'_n = g_1^{\alpha_{n1}} \cdot g_2^{\alpha_{n2}} \cdots g_n^{\alpha_{nn}} \cdots g_1^{\zeta_{n1}} \cdot g_2^{\zeta_{n2}} \cdots g_n^{\zeta_{nn}} \end{array}$$

and inquire under what circumstances they can serve as a set of generators for G. A necessary and sufficient condition for this is obviously that with the help of the generating relations (1) it shall be possible to solve the equations (10) so as to express $g_1, g_2, \cdots, g_n$ by equations analogous to (10) in terms of $g'_1, g'_2, \cdots, g'_n$. But it is not at all clear what conditions must be satisfied in order that this solution can be carried out.

The equations (10) determine an analogous set of equations for the commutative group $\tilde{G}$

$$(11) \qquad g'_i = g_1^{\mu_{i1}} g_2^{\mu_{i2}} \cdots g_n^{\mu_{in}} \qquad (i = 1, 2, \cdots, n)$$

in which

$$\mu_{ij} = \alpha_{ij} + \beta_{ij} + \cdots + \zeta_{ij}.$$

A solution of the equations (10) must correspond to a solution of the equations (11). But since the elements in (11) are commutative the process of solution is entirely analogous to that of solving the linear equations,

$$x'_i = \mu_{i1} x_1 + \mu_{i2} x_2 + \cdots + \mu_{in} x_n \quad (i = 1, 2, \cdots, n)$$

in terms of integers. A condition that a unique solution in integers shall exist is

$$
(12) \qquad \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{vmatrix} = \pm 1.
$$

31. If G is to be expressed in terms of the generators $g_1', g_2', \cdots g_n'$, the expressions for $g_1, g_2, \cdots, g_n$ in terms of $g_1', g_2', \cdots, g_n'$ must be substituted in the generating relations (1) in order to obtain a new set of generating relations in terms of $g_1', g_2', \cdots, g_n'$. When this set of generating relations is modified by allowing all elements to be commutative it becomes a new set of generating relations for $\tilde{G}$. This set of generating relations for $\tilde{G}$ could also be obtained by substituting directly the solutions of (11), if the latter exist. *This amounts to multiplying the matrix* $\|\gamma_{rs}\|$ *on the right by a square matrix of* n *rows and determinant* ± 1.

The generating relations (1) can also be modified by replacing them by equivalent expressions resulting from algebraic combinations. As applied to the matrix $\|\gamma_{rs}\|$ this means *multiplying it on the left by a square matrix of* k *rows and determinant* ± 1.

The two operations on $\|\gamma_{rs}\|$ are the operations required to reduce a matrix to the normal form E^* given in § 48, Chap. I, in which all elements are zero except the first r elements of the main diagonal which are denoted by $d_1, d_2, \cdots, d_r$. This reduction of $\|\gamma_{rs}\|$ to normal form determines one or more transformations of the generators and generating relations of $\tilde{G}$ to such a form that it has n generators subject to r generating relations,

$$
(13) \qquad \begin{aligned} g_1^{d_1} &= 1 \\ g_2^{d_2} &= 1 \\ &\cdot \\ &\cdot \\ &\cdot \\ g_r^{d_r} &= 1. \end{aligned}
$$

In case certain of the numbers $d_1, d_2, \cdots, d_r$ are 1, the corresponding generators of $\tilde{G}$ will be equal to the identity. In this case the symbols for these generators may be omitted, for $\tilde{G}$ is unaffected by introducing or removing a generator g_i which satisfies the condition $g_i = 1$.

32. Those of the numbers $d_1, d_2, \cdots, d_r$ which are not equal to 1 have been called by H. Tietze* the *Poincaré numbers* of the discrete group G. *They are invariants of* G *under all transformations to new sets of generators.* This is proved as follows.

An operation of $\tilde{G}$ is of finite period if and only if it is in the group H generated by the operations $g_1, g_2, \cdots, g_r$ and the relations (13). Hence H contains all operations of $\tilde{G}$ of finite period. Moreover H is a finite group because it is commutative and generated by a finite number of operations each of finite period. The invariant factors of $|\gamma_{it}|$ are invariants of H. This follows from §§ 35–37, Chap. IV.

These invariant factors are invariants of G because G determines the commutative group $\tilde{G}$ uniquely and $\tilde{G}$ determines the finite group H uniquely and H determines the invariant factors uniquely. Hence the transformations of the generators and generating relations of G do not change the Poincaré numbers.

33. If G is the fundamental group of a complex C_n it is evident from § 13 that G is the fundamental group of the two-dimensional complex composed of the 0-, 1-, and 2-cells of any regular subdivision of C_n. Hence the Poincaré numbers of G are the invariant factors of the matrix E_2 for this regular subdivision. Hence *they are the one-dimensional coefficients of torsion of* C_n.

Whether there exist generalizations of the fundamental group, and whether, in particular, these generalizations can be made in such a way as to bear a relation like the one just described to the n-dimensional Betti numbers and coefficients of torsion is a problem on which nothing has yet been published.

---

* Monatshefte für Math. u. Physik, Vol. 19 (1908), p. 56.

34. The equality of the Poincaré numbers of two discrete groups, G and G', is a necessary condition that the two groups be isomorphic, but it is far from being a sufficient condition. In fact, the problem of determining by a finite number of steps whether two groups, each given by means of a set of generators and a set of generating relations, are or are not isomorphic, seems to be a very difficult one. A clear discussion of this problem as well as of the general theory of discrete groups is given by M. Dehn, Math. Ann., Vol. 71 (1912), p. 116. The isomorphism problem has been solved for the following special case by J. Nielsen, Math. Ann., Vol. 79 (1919), p. 269: Let G be generated by n operations $g_1, g_2, \cdots, g_n$ subject to no relations, and let $g'_1, g'_2, \cdots, g'_n$ be a set of n operations of G which are subject to no relation; to determine by a finite number of steps whether the second set of operations also generate G. Further accounts of the theory of discrete groups, particularly the groups of two-dimensional manifolds, are to be found in the Kiel dissertation of H. Gieseking.

35. In an earlier paper (Math. Ann., Vol. 78, p. 385), Nielsen solves the isomorphism problem for the case $n = 2$ and applies the results to the study of systems of curves on the anchor ring. In the Math. Ann., Vol. 82 (1920), p. 83, he obtains a formula for the minimum number of fixed points of a homeomorphism of a two-dimensional manifold of genus 1 with itself in terms of the type of the homeomorphism, the type being determined by the isomorphism of the fundamental group which is effected by the given homeomorphism of the manifold. This is one of the papers referred to in § 7.

Covering Manifolds

36. The fundamental group of a complex C_n determines a covering manifold in the following manner. Let O be an arbitrary fixed point of C_n and let X be a general point. If σ^1 be any oriented 1-cell joining O to X it determines an infinite set of oriented 1-cell joining O to X which is such

that any oriented 1-cell of the set can be deformed into σ^1 through a set of intermediate positions all of which are oriented 1-cells joining O to X. The oriented 1-cells of such a set are said to be *equivalent* to one another. If g is any operation of the fundamental group, which is distinct from the identity, $g \cdot \sigma^1$ is not equivalent to σ^1. The set of all non-equivalent oriented 1-cells joining O to X may be represented in the form $g \cdot \sigma^1$ where σ^1 is fixed and g may be any operation of the fundamental group.

A set of points $[Y]$ on C_n may be defined by the convention that each Y is an X associated with the set of all oriented curves equivalent to a certain oriented 1-cell joining O to X. Thus for each X there is a set of Y's which is in (1–1) correspondence with the operations of the fundamental group. A 1-cell a^1 composed of X's determines a set of 1-cells composed of Y's each of which covers a^1, and there is one such 1-cell covering a^1 for each operation of the fundamental group. Similarly a k-cell a^k $(k = 0, 1, \cdots, n)$ composed of X's determines a set of k-cells composed of Y's, each such k-cell covering a^k, and the totality of k-cells which cover a^k being in (1 1) correspondence with the operations of the fundamental group.

37. In case $n = 2$ and C_2 is a complex determining a sphere, the set of points $[Y]$ is evidently a sphere covering C_2 once, because the fundamental group of C_2 is the identity. In case C_2 is a projective plane, $[Y]$ is a sphere covering C_2 twice. This follows because the group of the projective plane is a cyclic group of order two. The set of points $[Y]$ is essentially the same as the two-sided covering surface of a projective plane considered in § 11.

In general $[Y]$ is a complex of an infinite number of cells. If C_2 defines an orientable manifold not a sphere or a projective plane, $[Y]$ is homeomorphic with a single 2-cell. For a proof of this theorem and for a further consideration of covering manifolds, the reader is referred to the book of H. Weyl, Die Idee der Riemannschen Fläche, Leipzig, 1913.

The set of points $[Y]$ is called a *universal covering surface* of C_2 in case C_2 defines a manifold. If C_2 is reduced to

a normal form as in § 65, Chap. II, so that C_2 has only one 2-cell, the 2-cells of $[\Upsilon]$ which cover it may be regarded as constituting a network of regular polygons of $4p$ sides each, in a Euclidean ($p = 1$) or non-Euclidean plane. This is therefore the point at which the well-known applications to Automorphic Functions and the Uniformization theory fit into our outline of Analysis Situs.

Three-dimensional Manifolds

38. In the case of a two-dimensional manifold the fundamental group determines the orientation and the connectivity and therefore, the manifold, completely. In the three-dimensional case, such invariants as are known can be derived from a consideration of the fundamental group. For it has been shown above that the one-dimensional coefficients of torsion can be obtained from the group and from §§ 25, 27 and 31 it follows that P_1 is equal to the difference between the number of generators and the rank of the matrix $\|\gamma_{ij}\|$. Also, by § 19, Chap. IV, $P_1 = P_2$ if the manifold is orientable and $P_1 = P_2 + 1$ if it is one-sided. Hence P_1, P_2, and the coefficients of torsion are all derivable from the fundamental group.

It is natural to ask whether the fundamental group is determined by P_1 and the coefficients of torsion. This question was answered in the negative by Poincaré, who showed that there are manifolds for which $P_1 = 1$ and the coefficients of torsion are absent and for which the group does not reduce to the identity. An infinite class of such manifold has been studied by M. Dehn, Math. Ann., Vol. 69 (1910), p. 137, and called by him the *Poincaré spaces*. The group of a Poincaré space may be either finite or infinite.

It has also been proved that a three-dimensional manifold is not fully determined by its fundamental group. This was established by J. W. Alexander (Trans. Am. Math. Soc., Vol. 20 (1919), p. 339) by setting up two non-homeomorphic three-dimensional manifolds which have the same group, the cyclic group of order 5.

39. The problem still remains unsolved, however, to determine whether there is any three-dimensional manifold other than the three-dimensional sphere the fundamental group of which reduces to the identity.

The group of the covering manifold determined for any manifold M_n by its fundamental group obviously reduces to the identity. Hence in case the covering manifold is closed, the solution of this problem has an important bearing on the study of a manifold by means of its fundamental group.

The problem may be not entirely unrelated to the problem of generalizing the Schoenflies theorem referred to in § 19, Chap. III. The latter theorem has not yet been proved (so far as known to the writer) even for the following special case: Given a non-singular and simply connected two-dimensional polyhedron in a Euclidean space; to prove that the interior region of this polyhedron is homeomorphic with the interior of a tetrahedron.*

The Heegaard Diagram

40. The most direct way of attacking the problem of classifying three-dimensional manifolds is to try to reduce them to normal form by a process analogous to that outlined in § 62, Chap. II. If we start with a non-singular complex C_3 defining an orientable manifold M_3 and perform a sequence of operations (1) of coalescing pairs of 3-cell which have a common 2-cell on their boundaries and (2) of shrinking to a point 1-cells which join distinct points, C_3 is reduced to a complex C_3' consisting of one 3-cell and one 0-cell and equal numbers, α, of 2-cells and 1-cells. Hence M_3 may be represented by means of the interior and boundary of a Euclidean sphere, the boundary being a map all the vertices of which represent the same point of M_3. The 2-cells of this map fall into α pairs each of which represents a single 2-cell of C_3'. The 1-cells of the map fall into α sets

* This case has been proved by J. W. Alexander, Proc. Nat. Acad. of Sci., Vol. 10 (1924), pp. 6–8. The problem for a polyhedron in Euclidean n-space is still unsolved (1930).

such that all 1-cells in the same set represent the same 1-cell of C_3'.

41. This representation of a manifold M_3 by means of a sphere has not yet proved as fruitful as the related *Heegaard diagram* which may be obtained as follows: Let the 0-cell of C_3' be enclosed by a small 3-cell containing it and let each of the 1-cells of C_3' be enclosed by a small tube containing it. Thus we obtain a three-dimensional open manifold L_3 bounded by a two-dimensional manifold M_2 consisting of a sphere with α handles. M_3 is orientable if and only if M_2 is orientable. In case M_3 is orientable L_3 can be represented as the interior of a sphere with handles having no knots or links in a Euclidean 3-space. (Cf. § 44.)

The 2-cells of C_3' meet M_2 in a system of α curves no two of which intersect and which bound a set of α 2-cells $a_1^2, a_2^2, \cdots, a_\alpha^2$ contained in the 2-cells of C_3'. The points of the 2-cells a_i^2 together with the points of the 3-cell of C_3' which are not in L_3 or M_2 constitute the interior of an open three-dimensional manifold N_3 bounded by M_2.

Thus M_3 consists of two open manifolds L_3 and N_3 which have a common boundary, M_2. It is clear that M_3 is fully determined if L_3, M_2 and the boundaries $c_1, c_2, \cdots, c_\alpha$ of the cells a_i^2 are given. For the manifold M_3 can be reconstructed by putting in 2-cells bounded by the curves $c_1, c_2, \cdots, c_\alpha$ and a 3-cell bounded by M_2 and these 2-cells, each counted twice.

The representation of a manifold by means of L_3, M_2, and $c_1, c_2, \cdots, c_\alpha$ is called the Heegaard diagram. It is due (in a form which generalizes to n dimensions) to P. Heegaard in his dissertation, Forstudier til en topologisk teori for de algebraiske fladers sammenhaeng, Copenhagen, 1898 (republished in the Bulletin de la Soc. Math. de France, Vol. 44 (1916), p. 161). It is also described very clearly by M. Dehn, Math. Ann., Vol. 69, p. 165. Dehn draws from it the corollary that any M_3 can be defined by a non-singular complex having four 3-cells.

42. The curves $c_1, c_2, \cdots, c_\alpha$ are the boundaries of a set of 2-cells which reduce N_3 to a single 3-cell. In like manner there is a set of α curves, $d_1, d_2, \cdots, d_\alpha$ no two of which

have a point in common and which bound a set of α 2-cells which reduce L_3 to a single 3-cell. Moreover M_2 and the two sets of curves fully determine M_3.

In fact, suppose we have a manifold M_2 of genus α and two sets of curves $c_1, c_2, \cdots, c_\alpha$ and $d_1, d_2, \cdots, d_\alpha$, each set being such that no two of its curves have a point in common and such, moreover, that by introducing α 2-cells each bounded by one of the curves, M_2 is converted into a complex which can bound a 3-cell if each of the α 2-cells is counted twice. Then if we introduce a set of 2-cells of this sort for the curves $c_1, c_2, \cdots, c_\alpha$ and a 3-cell bounded by the resulting complex we obtain an open manifold L_3 bounded by M_2. If now we introduce another set of α 2-cells bounded by $d_1, d_2, \cdots, d_\alpha$ and having no points in common with each other or with L_3, or M_2, we can introduce another 3-cell bounded by M_2 and these 2-cells. The resulting three-dimensional complex is clearly a manifold which is homeomorphic with M_3 if M_2 and the curves were determined from M_3 in the manner described in the paragraph above.

43. The problem of three-dimensional manifolds is thus reduced to one regarding systems of curves upon a two-dimensional manifold. The modifications which can be made in the systems of curves of a Heegaard diagram without changing the manifold M_3 represented by the diagram have been studied (though not completely) by Heegard in his dissertation. The most important results thus far obtained on systems of curves are those of Poincaré in his fifth complement, in which he was evidently considering the problem of three-dimensional manifolds from approximately the point of view outlined in the last section. Reference should also be made in connection with the problem of systems of curves on two-dimensional manifolds to two articles by C. Jordan in the Journal de Mathematique, Ser. 2, Vol. 11 (1866), to an article by Dehn, Math. Ann., Vol. 72 (1912), to the dissertation of J. Nielsen, Kiel, 1913, to the article by Brahana cited in § 65, Chap. II, and to the articles on the group of a two-dimensional manifold already referred to in this chapter.

The Knot Problem

44. Very closely related to the problem of classifying the three-dimensional manifolds is the problem of classifying the knots in the three-dimensional Euclidean or spherical space. A *knot* may be defined as a non-singular curve in a Euclidean space which is not isotopic with the boundary of a triangular region; and two knots are regarded as of the same type if and only if they are isotopic.

A large number of types of knots have been described by Tait and others and a list of references will be found in the Encyklopädie article on Analysis Situs. But a more important step towards developing a theory of knots was taken by M. Dehn, who introduced the notion of the group of the knot, which is essentially the group of the generalized three-dimensional complex obtained by leaving out the knot from the three-dimensional space. Dehn gave a method for obtaining the group of a knot explicitly and applied it to the construction of the Poincaré spaces already referred to (§ 38). Dehn's work is to be found in his articles in the Math. Ann. in Vols. 69 and 71 to which we have already referred and in an article on the two trefoil knots in Vol. 75 (1914), p. 402.

It is obvious that if a three-dimensional Riemann space of k sheets be found which has a given knot as its only branch curve, the invariants (Betti numbers, etc.) of this space will be invariants of the given knot. This method of studying the invariants of a knot has been deleloped by J. W. Alexander in a paper read before the National Academy of Sciences in November 1920, but not yet published.*

* (1930.) See a paper by J. W. Alexander and G. B. Briggs, Annals of Math. (2), Vol. 28 (1927), pp. 562-586, and a paper by J. W. Alexander, Trans. Amer. Math. Soc., Vol. 30 (1928), pp. 275-306.

APPENDIX 1*

THE INTERSECTION NUMBERS†

1. In his first memoir on analysis situs‡ Poincaré defined a number $N(\Gamma_k, \Gamma_{n-k})$ which had previously been considered, at least in special cases, by Kronecker. With certain conventions as to sign this number represents the excess of the number of positive over the number of negative intersections of a k-dimensional circuit Γ_k with an $(n-k)$-dimensional circuit Γ_{n-k} when both are immersed in an n-dimensional oriented manifold. The purpose of the present paper is to show how to calculate this number when the manifold is defined combinatorially as a collection of cells and the circuits are composed of sets of these cells; and to show how the matrices which represent the intersectional relations between the k-circuits and the $(n-k)$-circuits depend on the matrices of orientation of the manifold. We also define certain modulo 2 intersection numbers and discuss the matrices connected with them.

2. Let a manifold M_n be given as the set of all points of a complex C_n. Let C_n' be a complex dual to C_n constructed as explained in § 25, Chap. III by means of a complex $\bar{C}_n$ which is a regular subdivision both of C_n and of C_n'. Every k-cell a_j^k of C_n has a single point P_j^k (cf. § 20, Chap. III) in common with a single $(n-k)$-cell of C_n' which is called b_j^{n-k}. Our first problem will be to assign a positive or negative sign to the intersection of a_j^k with b_j^{n-k}.

* Reprinted, with minor changes, from the Transactions of the American Mathematical Society, Vol. 25 (1923), pp. 540–550.

† Presented to the Society under a different title, April 24, 1920.

‡ Journal de l'École Polytechnique, ser. 2, vol. 1 (1895).

In order to do this, we suppose M_n to be oriented as explained in Chapter IV and that all cells, circuits, etc., are oriented. Moreover, in the regular complex $\bar{C}_n$, in which each i-cell is uniquely determined by its $i+1$ vertices, the orientation of the i-cell will be denoted by the order in which its vertices are written, and the following two conventions will be followed: (1) if $A_0 A_1 \cdots A_k$ denotes a given oriented k-cell ($k = 1, 2, \cdots, n$) any even permutation of $A_0 A_1 \cdots A_k$ denotes the same oriented k-cell and any odd permutation denotes its negative; (2) the oriented $(k-1)$-cell $A_1 A_2 \cdots A_k$ is positively related to the oriented k-cell $A_0 A_1 \cdots A_k$.

A simple argument by mathematical induction could, but will not here, be given to prove that these notations and conventions are consistent with themselves and with the definition of oriented cells.

3. The k-cell a_j^k of C_n is made up of a number of k-cells of $\bar{C}_n$ having P_j^k as their common vertex. Using the notation of § 20, Chap. III, let one of these be denoted by

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k,$$

the points P being chosen, as is always possible, so that the orientation of this k-cell agrees with that of a_j^k. In like manner, b_j^{n-k} is made up of a number of $(n-k)$-cells of C_n having P_j^k as their common vertex, and we let any one of these be denoted by

$$P_j^k P_l^{k+1} \cdots P_s^n,$$

the points P being chosen this time so that the sense of the k-cell which they represent agrees with that of b_j^{n-k}. According as the oriented n-cell

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$$

is positively or negatively oriented, we say that the intersection of a_j^k with b_j^{n-k} is positive or negative. In the first case we write

$$N(a_j^k, b_j^{n-k}) = 1$$

and in the second case

$$N(a_j^k, b_j^{n-k}) = -1.$$

From the definition of the points P it follows directly that this definition is independent of the particular cells of $\bar{C}_n$ which it employs. It also follows that the function N is such that

$$N(a_j^k, b_j^{n-k}) = -N(-a_j^k, b_j^{n-k}) \tag{3.1}$$
$$= -N(a_j^k, -b_j^{n-k}).$$

Since the relation between C_n and C_n' is reciprocal, the definition given here determines the meaning of $N(b_j^{n-k}, a_j^k)$, and a count of trauspositions in the notation with other simple considerations gives the formula

$$N(b_j^{n-k}, a_j^k) = (-1)^{k(n-k)} N(a_j^k, b_j^{n-k}). \tag{3.2}$$

4. The cells of C_n and C_n' are so oriented (cf. § 38, Chap. IV) that

$$E_k' = E_{n-k+1},$$

which means that a_j^k is positively or negatively related to a_i^{k-1} according as b_i^{n-k+1} is positively or negatively related to b_j^{n-k}. Now the points P may be so chosen that $P_a^0 P_b^1 \cdots P_i^{k-1}$ represents an oriented cell on a_i^{k-1} and $P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$ represents an oriented cell on b_i^{n-k+1}. By the definition in § 2 above, the oriented cell $P_a^0 P_b^1 \cdots P_i^{k-1}$ is positively or negatively related to $P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k$, and therefore to a_i^k, according as $(-1)^k$ is positive or negative. On the other hand, $P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$ is positively related to $P_j^k P_l^{k+1} \cdots P_s^n$, and therefore to b_i^{n-k}. Hence if b_i^{n-k+1} is positively related to b_i^{n-k}, a_i^{k-1} is positively related to a_i^k and $(-1)^k N(a_i^{k-1}, b_i^{n-k+1})$ is positive or negative according as

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$$

is positively or negatively oriented. A similar result holds if b_i^{n-k+1} is negatively related to b_j^{n-k}. Hence

$$N(a_j^k, b_j^{n-k}) = (-1)^k N(a_i^{k-1}, b_i^{n-k+1}).$$

By repeated application of this formula we obtain

$$N(a_j^k, b_j^{n-k}) = (-1)^{k(k+1)/2} N(a_a^0, b_a^n).$$

But all the n-cells b_i^n are similarly oriented. Hence the value of $N(a_a^0, b_a^n)$ is the same for all 0-cells a_a^0, and consequently the value of $N(a_j^k, b_j^{n-k})$ is independent of j. Hence if the notation is so chosen that b_i^n is positively oriented,*

$$\begin{aligned} N(a_i^0, b_i^n) &= 1, \\ N(a_i^1, b_i^{n-1}) &= -1, \\ N(a_i^2, b_i^{n-2}) &= -1, \\ N(a_i^3, b_i^{n-3}) &= 1, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

and all these equations are independent of i.

5. An oriented complex Γ_k containing $\overset{1}{x}, \overset{2}{x}, \cdots, \overset{\alpha_k}{x}$ oriented k-cells coincident with $a_1^k, a_2^k, \cdots, a_{\alpha_k}^k$, respectively, is represented by the notation

$$\Gamma_k = (\overset{1}{x}, \overset{2}{x}, \cdots, \overset{\alpha_k}{x}). \tag{5.1}$$

Let Γ'_{n-k} be an arbitrary oriented complex of C'_k, so that

$$\Gamma'_{n-k} = (\overset{1}{y}, \overset{2}{y}, \cdots, \overset{\alpha_k}{y}). \tag{5.2}$$

By the number of intersections of Γ_k with Γ'_{n-k}, having regard to sign, we shall mean the number $N(\Gamma_k, \Gamma'_{n-k})$ defined by means of the equation

$$\begin{aligned} N(\Gamma_k, \Gamma'_{n-k}) &= \sum_{j=1}^{\alpha_k} \overset{j}{x} \overset{j}{y} N(a_j^k b_j^{n-k}) \\ &= (-1)^{k(k+1)/2} \sum_{j=1}^{\alpha_k} \overset{j}{x} \overset{j}{y}. \end{aligned} \tag{5.3}$$

* Cf. Poincaré, Proceedings of the London Mathematical Society, vol. 32 (1900), p. 280.

(If $N = 0$, we say that Γ_k and Γ'_{n-k} do not intersect.) If we recall that there are no intersections of cells of Γ_k of dimensionality less than k with cells of Γ'_{n-k} and that no cell a_i^k intersects a cell b_j^{n-k} unless $i = j$, it is clear that this definition is in accordance with geometric intuition.

6. The last equation has as obvious corollaries the equations

$$N(\Gamma_k + \Delta_k, \Gamma'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Delta_k, \Gamma'_{n-k}), \tag{6.1}$$

$$N(\Gamma_k, \Gamma'_{n-k} + \Delta'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Gamma_k, \Delta'_{n-k}), \tag{6.2}$$

from which it follows that if Γ_k^i $(i = 1, 2, \cdots, \alpha_k)$ is any set of k-dimensional complexes of which all k-dimensional complexes of C_n are linear combinations and $\Gamma_{n-k}^{i'}$ $(i = 1, 2, \cdots, \alpha_k)$ a set of $(n-k)$-dimensional complexes of which all $(n-k)$-dimensional complexes of C'_n are linear combinations, then if

$$\Gamma_k = \sum_{i=1}^{\alpha_k} x_i \Gamma_k^i \tag{6.3}$$

and

$$\Gamma'_{n-k} = \sum_{i=1}^{\alpha_k} y_j \Gamma_{n-k}^{j'} \tag{6.4}$$

where the x's and y's are integers, then

$$N(\Gamma_k, \Gamma'_{n-k}) = \sum_{i=1}^{\alpha_k} \sum_{j=1}^{\alpha_k} x_i y_j N(\Gamma_k^i, \Gamma_{n-k}^{j'}). \tag{6.5}$$

Hence the intersection numbers of all k-dimensional complexes of C_n with all $(n-k)$-dimensional complexes of C'_n depend on the matrix of numbers $N(\Gamma_k^i, \Gamma_{n-k}^{j'})$. By choosing the complexes Γ_k^i and $\Gamma_{n-k}^{j'}$ in the normal manner described in Chap. IV this matrix may be given a very simple form, which we shall determine in the next three sections.

7. As proved in §§ 15, 16, Chap. IV, a set of k-dimensional complexes upon which all the complexes formed from cells of C_n are linearly dependent may be so chosen as to consist of (1) a set of $P_k - 1$ non-bounding sets of circuits which we shall denote by Γ_k^i $(i = 1, \cdots, P_k - 1)$, or in Poincaré's notation,

$$\Gamma_k^i \equiv 0; \tag{7.1}$$

(2) a set of τ_k sets of circuits A_k^i $(i = 1, \cdots, \tau_k)$ which satisfy the homologies

$$(7.2) \qquad t_i^k A_k^i \sim 0$$

in which t_i^k represents a k-dimensional coefficient of torsion; (3) a set of $r_{k+1} - \tau_k$ bounding sets of circuits Θ_k^i

$$(7.3) \qquad \Theta_k^i \sim 0;$$

and (4) and (5) two sets of complexes Φ_k^i and Ψ_k^i which are not sets of circuits but satisfy the following congruences:

$$(7.4) \qquad \Phi_k^i \equiv \Theta_{k-1}^i, \qquad 0 < i < r_k - \tau_{k-1},$$

$$(7.5) \qquad \Psi_k^i \equiv t_i^{k-1} A_{k-1}^i, \qquad 0 < i < \tau_{k-1},$$

in which Θ_{k-1}^i and A_{k-1}^i are defined by replacing k by $k-1$ in (7.3) and (7.2).

These relations are derived from the matrix equation

$$(7.6) \qquad E_k \cdot D_k = C_{k-1} \cdot E_k^*$$

which arises in reducing (cf. §§ 14–16, Chap. IV) the orientation matrix E_k to normal form. The matrix E_k^* is one in which all elements are zero except the first r_k elements of the main diagonal. The first $r_k - \tau_{k-1}$ of the non-zero elements are 1 and the remaining τ_{k-1} are the coefficients of torsion of dimensionality $k-1$.

The first $r_k - \tau_{k-1}$ columns of D_k represent the complexes Φ_k^i, the next τ_{k-1} columns represent the complexes Ψ_k^i, the next $P_k - 1$ columns represent the sets of circuits Γ_k^i, the next $r_{k+1} - \tau_k$ columns represent the sets of circuits Θ_k^i, the next τ_k columns represent the sets of circuits A_k^i. Thus, for example, if the jth column of D_k $(0 < j < r_k - \tau_{k-1})$ is $(x_{1j}, x_{2j}, \cdots, x_{\alpha_k j})$ we have

$$(7.7) \qquad \Phi_k^j = (x_{1j}, x_{2j}, \cdots, x_{\alpha_k j}).$$

The columns of the matrix C_{k-1} are the same as the columns of D_{k-1} in a different order, and each complex represented

by a column of D_k is bounded by the circuit or set of circuits represented by the corresponding column of the matrix $C_{k-1} \cdot E_k^*$. It is from this fact that the congruences (7.4) and (7.5) are derived. The fact that Γ_k^i, Λ_k^i, Θ_k^i are sets of circuits is a consequence of the fact that all elements of E_k^* subsequent to the r_kth column are zero.

The homologies (7.2) and (7.3) arise by similar reasoning from the matrix equation

$$E_{k+1} \cdot D_{k+1} = C_k \cdot E_{k+1}^* \tag{7.8}$$

in which it is to be remembered that the columns of C_k are the same as those of D_k in a different order.

8. The $(n-k)$-dimensional complexes required in the formulas of § 6 may be determined by the same process as described in § 7, from the matrices of the dual complex C_n'. The matrices of the dual complex are related to those of C_n by the equation (cf. § 38, Chap. IV)

$$\bar{E}_{n-k} = E_{k+1}' \tag{8.1}$$

in which $\bar{E}_{n-k}$ is the matrix of the relations between $(n-k-1)$-cells and $(n-k)$-cells of C_n' and E_{k+1}' is the matrix obtained by interchanging rows and columns of E_{k+1}. The equation (7.8) gives the following:

$$\begin{aligned} C_k^{-1} \cdot E_{k+1} \cdot D_{k+1} &= E_{k+1}^*, \\ D_{k+1}' \cdot E_{k+1}' \cdot C_k^{-1'} &= E_{k+1}^{*'}, \\ \bar{E}_{n-k} \cdot C_k^{-1'} &= D_{k+1}^{-1'} \cdot E_{k+1}^{*'}. \end{aligned}$$

The columns of $C_k^{-1'}$ determine a linearly independent set of complexes analogous to those determined by the columns of D_k. They are described by the following homologies and congruences, written in the order of the columns of $C_k^{-1'}$:

$$\bar{\Phi}_{n-k}^j \equiv (-1)^{\alpha_{k+1}-r_{k+1}} \bar{\Theta}_{n-k-1}^j, \quad 0 < j \leq r_{k+1} - \tau_k; \tag{8.2}$$

$$\bar{\Psi}_{n-k}^j \equiv (-1)^{\alpha_{k+1}-r_{k+1}} \cdot t_j^k \bar{\Lambda}_{n-k-1}^j, \qquad 0 < j \leq \tau_k; \tag{8.3}$$

$$(8.4)\qquad \Gamma^j_{n-k} \equiv 0, \qquad 0 < j \leqq P_k - 1;$$

$$(8.5)\qquad \bar{\Theta}^j_{n-k} \sim 0, \qquad 0 < j \leqq r_k - \tau_{k-1};$$

$$(8.6)\qquad t_j^{k-1}\,\bar{\Lambda}^j_{n-k} \sim 0; \qquad 0 < j \leqq \tau_{k-1}.$$

9. Since the columns of $C_k^{-1'}$ are the same as the rows of C_k^{-1}, the matrix equation

$$(9.1)\qquad C_k^{-1} \cdot C_k = 1$$

implies the relations

$$(9.2)\qquad \sum_{i=1}^{\alpha_k} x_{ij}\, x'_{ip} = \begin{cases} 1 \text{ if } j = p \\ 0 \text{ if } j \neq p \end{cases}$$

between the columns $(x_{1j}, x_{2j}, \cdots, x_{\alpha_k j})$ of C_k and the columns $(x'_{1p}, x'_{2p}, \cdots, x'_{\alpha_k p})$ of $C_k^{-1'}$. But by (5.3), and using the fact that the columns of C_k are obtained in a particular way from those of D_k (§ 16, Chap. IV), this implies that the intersection numbers of Γ^j_k, Λ^j_k, etc., with $\bar{\Gamma}^j_{n-k}$, $\bar{\Lambda}^j_{n-k}$, etc., are zero except in the following α_k cases, written in the order of the columns of $C_k^{-1'}$:

$$(9.3)\qquad N(\Theta^j_k, \bar{\Phi}^j_{n-k}) = (-1)^{k(k+1)/2}, \quad 0 < j \leqq r_{k+1} - \tau_k;$$

$$(9.4)\qquad N(\Lambda^j_k, \bar{\Psi}^j_{n-k}) = (-1)^{k(k+1)/2}, \quad 0 < j \leqq \tau_k;$$

$$(9.5)\qquad N(\Gamma^j_k, \bar{\Gamma}^j_{n-k}) = (-1)^{k(k+1)/2}, \quad 0 < j \leqq P_k - 1;$$

$$(9.6)\qquad N(\Phi^j_k, \bar{\Theta}^j_{n-k}) = (-1)^{k(k+1)/2}, \quad 0 < j \leqq r_k - \tau_{k-1};$$

$$(9.7)\qquad N(\Psi^j_k, \bar{\Lambda}^j_{n-k}) = (-1)^{k(k+1)/2}, \quad 0 < j \leqq \tau_{k-1}.$$

Thus, each set of k-circuits Γ^i_k intersects the corresponding set of $(n-k)$-circuits once and intersects no other of the fundamental $(n-k)$-dimensional complexes. None of the other sets of k-circuits (Λ^i_k or Θ^i_k) intersects any $(n-k)$-circuits, but each Θ^i_k intersects a complex $\bar{\Phi}^i_{n-k}$ which is bounded by $(-1)^{\alpha_{k+1}-r_{k+1}}\,\bar{\Theta}^i_{n-k-1}$; and each Λ^i_k intersects a complex $\bar{\Psi}^i_{n-k}$ which is bounded by $(-1)^{\alpha_{k+1}-r_{k+1}}\,\bar{\Lambda}^i_{n-k-1}$ taken t^k_i times.

Thus we may say that each Θ_k^i *links* one and only one $\bar{\Theta}_{n-k-1}^i$ once and each A_k^i links one $\bar{A}_{n-k-1}^i$ in a manner which may be described as a fractional number of times, $\pm 1/t_i^k$. A further study of these linkages would carry us beyond the bounds of the present paper.

10. The matrix spoken of at the end of § 6 is now seen to consist entirely of zeros except for α_k elements whose value, ± 1 in every case, is given by equations (9.3), $\cdots$, (9.7). If we limit attention to sets of circuits the only non-zero terms which remain are those given by the intersections of Γ_k', $\cdots$, $\Gamma_k^{P_k-1}$ with the corresponding non-bounding sets of $(n-k)$-circuits. The matrix is therefore one which consists entirely of zeros except for the first $P_k - 1$ terms of the main diagonal which are all 1's. For any set of k-circuits Γ_k of C_n we have

$$(10.1) \qquad \Gamma_k = \sum_{i=1}^{P_k-1} x_i \Gamma_k^i + \sum_{i=1}^{\tau_k} y_i A_k^i + \sum_{i=1}^{r_{k+1}-\tau_k} z_i \Theta_k^i$$

and for any set of $(n-k)$-circuits of C_n' we have

$$(10.2) \quad \bar{\Gamma}_{n-k} = \sum_{i=1}^{P_k-1} x_i' \bar{\Gamma}_{n-k}^i + \sum_{i=1}^{\tau_{k-1}} y_i' \bar{A}_{n-k}^i + \sum_{i=1}^{r_k-\tau_{k-1}} z_i' \bar{\Theta}_{n-k}^i .$$

When these expressions are substituted in (6.5) there results

$$(10.3) \qquad N(\Gamma_k, \bar{\Gamma}_{n-k}) = (-1)^{k(k+1)/2} \sum_{i=1}^{P_k-1} x_i x_i' .$$

Thus we have the theorem that *if*

$$(10.4) \qquad \Gamma_k \sim \sum_{i=1}^{P_k-1} x_i \Gamma_k^i + \sum_{i=1}^{\tau_k} y_i A_k^i$$

and

$$(10.5) \qquad \bar{\Gamma}_{n-k} \sim \sum_{i=1}^{P_k-1} x_i' \bar{\Gamma}_{n-k}^i + \sum_{i=1}^{\tau_{k-1}} y_i' \bar{A}_{n-k}^i ,$$

then the intersection number of Γ_k with $\bar{\Gamma}_{n-k}$ is given by (10.3).

This theorem has the corollary that $p\Gamma_k \sim 0$ for some $p \neq 0$, is equivalent to the equation

$$N(\Gamma_k, \bar{\Gamma}_{n-k}) = 0$$

for the one set of circuits Γ_k and all sets of circuits $\overline{\Gamma}_{n-k}$.

From this it follows that if Γ'_k is any set of k-circuits composed of cells coincident with cells of C_n and such that for $p \neq 0$

$$p\,\Gamma_k \sim p\,\Gamma'_k$$

then

$$N(\Gamma_k, \overline{\Gamma}_{n-k}) = N(\Gamma'_k, \overline{\Gamma}_{n-k}).$$

11. Incidentally it may be remarked that (10.4) and (10.5) give rise to the following "homologies with division allowed":

$$\Gamma_k \sim \sum_{i=1}^{P_k-1} x_i\, \Gamma^i_k, \qquad \overline{\Gamma}_{n-k} \sim \sum_{i=1}^{P_k-1} x'_i\, \overline{\Gamma}^i_{n-k}.$$

Whenever these homologies are satisfied the equation (10.3) is satisfied. As remarked by Poincaré, it is because the intersection numbers are more closely related to the homologies with division allowed than to the ordinary homologies that his attempt to prove the Euler theorem and the theorem about the duality of the Betti numbers by means of the intersection numbers was unsuccessful.

12. The fundamental sets of circuits which appear in the formulas of § 10 are chosen in a very special manner. A perfectly arbitrary fundamental set of non-bounding sets of k-circuits is however related to this special set by homologies with division allowed

$$G^i_k \sim \sum_{j=1}^{P_k-1} \alpha^i_j\, \Gamma^j_k$$

in which the $(P_k - 1)$-rowed determinant $|\alpha^i_j| = \pm 1$. A general fundamental set of non-bounding sets of $(n-k)$-circuits $\overline{G}^i_{n-k}$ is related to the special set by an analogous set of homologies. Hence the matrix of the intersection numbers

$$N(G^i_k, \overline{G}^j_{n-k})$$

is one of $P_k - 1$ rows and $P_k - 1$ columns, of rank $P_k - 1$ and having all its invariant factors unity.

13. For some purposes it is desirable to introduce intersection numbers which do not distinguish between positive

and negative intersections. The theory of these numbers is much simpler than that which we have been developing because all the determinations of algebraic sign in §§ 2, 3, 4, 5, 8 can be omitted. We simply replace the definitions of § 3 by the agreement that

$$M(a_i^k,\, b_j^{n-k}) = 1 \text{ or } 0$$

according as a_i^k and b_j^{n-k} have a common point or not. Then the definition in § 5 is replaced by

$$M\,(\Gamma_k,\, \Gamma'_{n-k}) = \sum_{j=1}^{\alpha_k} x^j y^j,$$

the sum being taken modulo 2.

The determination of the intersection numbers of fundamental sets of k-circuits and $(n-k)$-circuits in §§ 7, 8, 9 is replaced by an analogous theory based on the matrices A_{k-1} and B_k which arise in the reduction of the incidence matrix H_k to normal form (cf. §§ 11–14, Chap. III). The result obtained is that there exist a set of k-circuits $\Gamma_k^1, \Gamma_k^2, \cdots, \Gamma_k^{R_k-1}$ and a set of $(n-k)$-circuits $\bar{\Gamma}_{n-k}^1, \bar{\Gamma}_{n-k}^2, \cdots, \bar{\Gamma}_{n-k}^{R_k-1}$ such that

$$M(\Gamma_k^i,\, \bar{\Gamma}_{n-k}^j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j; \end{cases}$$

and if

$$\Gamma_k \sim \sum_{i=1}^{R_k-1} x_i \Gamma_k^i,$$

$$\Gamma_{n-k} \sim \sum_{i=1}^{R_k-1} y_i \bar{\Gamma}_{n-k}^i,$$

then

$$M(\Gamma_k,\, \bar{\Gamma}_{n-k}) = \sum_{i=1}^{R_k-1} x_i y_i \pmod 2.$$

It should be observed that these formulas cannot be obtained by reducing the formulas of § 10, modulo 2, because the formulas of the present section take account of non-orientable sets of circuits which do not enter into the theory of oriented intersections.

APPENDIX II*

ON MATRICES WHOSE ELEMENTS ARE INTEGERS.

BY OSWALD VEBLEN AND PHILIP FRANKLIN.

Introduction.

1. The purpose of this article is strictly expository. The aim is to set forth some of the theorems on matrices whose elements are integers. These theorems have applications in Analysis Situs and the systematic treatment of them directly in terms of integers here given will no doubt be useful to students of that subject. While the closely allied algebraic theory is to be found in Bôcher's Introduction to Higher Algebra, and the matter here given is to some extent discussed in Muth's Elementartheiler and in Scott and Mathews' Determinants, there is no readily accessible treatment of the subject from the point of view here adopted.

2. The object of our study will be a matrix of α rows and β columns:

$$(1) \qquad E = \|\varepsilon_i^j\| = \left\| \begin{matrix} \varepsilon_1^1 & \varepsilon_1^2 & \cdots & \varepsilon_1^\beta \\ \varepsilon_2^1 & \varepsilon_2^2 & \cdots & \varepsilon_2^\beta \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \varepsilon_\alpha^1 & \varepsilon_\alpha^2 & \cdots & \varepsilon_\alpha^\beta \end{matrix} \right\|.$$

The elements of E are integers. The term "integer" here includes negative integers and zero; but we shall assume that at least one element is different from zero.

* Reprinted, with minor changes, from Annals of Mathematics (2), Vol. 23 (1921), pp. 1–15.

Our definition of the *product* of two matrices $\|\varepsilon_i^j\|$ and $\|\eta_i^j\|$ is:

(2) $$\|p_i^j\| = \|\varepsilon_i^j\| \cdot \|\eta_i^j\|,$$

where

(3) $$p_i^j = \sum_{k=1}^{k=\beta} \varepsilon_i^k \cdot \eta_k^j.$$

The number of rows of the second matrix must be equal to the number of columns of the first; and the product has as many rows as the first matrix and as many columns as the second. If the matrices are square, the product will be square, and the determinant of the product will be equal to the product of the determinants of the factors.

The *inverse* of a square matrix A, of determinant unity, will be the matrix A^{-1} such that:

$$A^{-1} \cdot A = A \cdot A^{-1} = I,$$

where I denotes the identity matrix $\|\delta_i^j\|$, a square matrix with all the elements in the main diagonal $+1$ and all the remaining elements zeros. The element $\breve{a}_i^j$ of A^{-1} will evidently be the cofactor of a_j^i in the determinant of A. A is restricted to be of determinant unity to insure the elements of the inverse matrix being integers. We might also admit the value -1, as is indeed found convenient in applications to Analysis Situs. One of the advantages of admitting both signs is pointed out in § 9 below.

Elementary Transformations

3. Let us consider two types of transformations of E:

(*a*) *To replace each element of the rth row* (ε_r^j) *by the element* $(\varepsilon_r^j + q\,\varepsilon_s^j)$ *where q is either* $+1$ *or* -1 *and* $s \neq r$. This operation is described as adding the sth row to the rth row or subtracting the sth row from the rth row.

(*b*) *To add a column to or subtract it from another column.*

The operation (*a*) is equivalent to multiplying E on the left by a square matrix of α rows $A_0 = \|a_i^j\|$ in which all the elements are zeros except those of the main diagonal

which are $+1$, and a_r^s which is q. For the expressions given by (3) for the elements of the product, i. e.,

$$(4) \qquad p_i^j = \sum a_i^k \cdot \varepsilon_k^j$$

reduce to the single term ε_i^j except when $i = r$; in which case they give the two terms:

$$\varepsilon_r^j + q\varepsilon_s^j.$$

That is, the operation (a) transforms E into $A_0 \cdot E$.

In like manner, the operation (b) corresponds to multiplying E on the right by a square matrix of β rows $B_0 = \|b_i^j\|$ in which all the elements are zeros except those of the main diagonal which are 1 and b_s^r which is q.

If the operation (a) be repeated n times, where n is a positive integer, the effect is an operation identical with (a) except that q is replaced by the integer n or $-n$. Correspondingly, the effect of multiplying the matrix A_0 by itself repeatedly is to change the element a_r^s to $\pm n$.

The inverse of A_0 if $a_r^s = \pm 1$ is the same matrix except that the sign of a_r^s is changed. Hence the inverse of an operation of type (a) is an operation of the same type. The determinant of A_0 is $+1$.

Similar statements hold with regard to the operation (b) and the matrix B_0.

4. *The operation of interchanging two rows of a matrix and changing the signs of all the elements of one of them can be expressed as a sequence of operations of type* (a). For if we add the rth row to the sth, then subtract the sth row of the resulting matrix from the rth, and finally add the rth row to the sth, the elements of the rth and sth rows (and the qth column) will be, successively:

$$(\varepsilon_r^q, \varepsilon_s^q); \quad (\varepsilon_r^q, \varepsilon_r^q + \varepsilon_s^q); \quad (-\varepsilon_s^q, \varepsilon_r^q + \varepsilon_s^q); \quad (-\varepsilon_s^q, \varepsilon_r^q);$$

and the resulting matrix will thus be that obtained by changing the signs of the elements of the sth row and then interchanging the rth and sth rows.

In like manner, *the operation of interchanging two columns and changing the signs of the elements of one of them is expressible as a sequence of operations of type* (b).

5. In place of our two fundamental operations (a) and (b) we might have restricted ourselves to the operations:

(a') *To add a row to, or subtract it from, an adjacent row.*

(b') *To add a column to, or subtract it from, an adjacent column.*

To prove this, we note that the proof in § 4 shows that we can interchange two adjacent rows and change the signs of the elements of one of them, by means of operations of type (a'). Let us call the former operation one of type (a''). Then by use of operations of type (a''), we can bring any two given rows to adjacent positions; then use (a') to add one to, or subtract it from, the other; and finally use operations of type (a'') to return the rows to their original positions. In applying the operations of type (a''), we always keep fixed the signs of the elements of the two rows on which we wish to perform the operation (a). As a result, all other rows end up in their original positions, with the signs of their elements unchanged.

As a similar argument holds for steps (b) and (b'), if we replace rows by columns, we conclude that *the transformations built up from steps* (a) *and* (b) *are no more general than those built up from steps* (a') *and* (b').

Determinant Factors

6. Consider the set of γ-rowed ($0<\gamma \leqq \alpha$, $\gamma \leqq \beta$) determinants which can be formed from E by omitting $\alpha-\gamma$ of the rows and $\beta-\gamma$ of the columns in all possible ways. The highest common factor (H. C. F.) of such a set of determinants, if the determinants are not all zero, is denoted by D_γ and is called the γth *determinant factor** of E.

The determinant factors are unchanged when the matrix is operated on by transformations of type (a) *or* (b). For, consider the effect of an operation of type (a) which consists in adding the rth row to (or subtracting it from) the sth,

* Cf. Scott and Mathews' Determinants, p. 76.

on the γ-rowed determinants in question. All such determinants which do not contain elements from the sth row are obviously unaffected, while those that contain elements from both the rth row and the sth are not affected because of an elementary theorem on determinants. The remaining γ-rowed determinants, as A_γ, which contain elements from the sth row and not from the rth, are converted into determinants of the form $A_\gamma \pm A'_\gamma$ where A'_γ is the γ-rowed determinant obtained from A_γ by replacing the elements from the sth row by elements from the same columns of the matrix and from the rth row. Since A'_γ equals plus or minus a determinant which can be formed both from the old matrix and from the new matrix, the H. C. F. is easily seen to be unchanged.

The proof for operations of type (*b*) is similar.

7. The following theorem has an application in Analysis Situs: *If a matrix E is such that each column either consists entirely of zeros or contains just two elements different from* 0, *one* $+1$ *and the other* -1, *all the determinant factors of the matrix are* $+1$ *or* -1.

The theorem follows immediately from the definition of a determinant factor, if we observe that any γ-rowed determinant formed by striking out $(\alpha-\gamma)$ rows and $(\beta-\gamma)$ columns of the given matrix has either two, none or one element in each column different from zero. If no column is of the third type the determinant is zero, since the sum of all the elements in each column is zero. If there is a column of the third type we evaluate the determinant with reference to such a column and then evaluate the minor with reference to a column with a single non-zero element in the minor, and so on. In this way we either arrive finally at ± 1 for the value of the determinant, or else come to a minor with two or no non-zero elements in each column, in which case the determinant is zero.

Reduction to Normal Form

8. Let us now consider a series of reductions of the matrix E which can be effected by transformations of types (*a*) and (*b*).

If the first column consists entirely of zeros, add one of the other columns to it. Thus by a transformation of type (b) E is converted into a matrix E_1 which has at least one non-zero element in the first column. If the first element of the first column is zero, add a row which contains a non-zero element in the first column to the first row. Thus by a transformation of type (a) E_1 is converted into a matrix E_2 for which the element of the first row and column is not zero.

We shall now prove that if this non-zero element ϵ_1^1 is not a factor of all the elements of the matrix, we can, by a series of transformations of types (a) and (b), replace it by a numerically smaller element different from zero.

First, if one of the elements in the first column, ϵ_r^1, is not divisible by ϵ_1^1, upon adding the first row to (or subtracting it from) the rth a number of times equal to the largest integer in the quotient of ϵ_r^1 by ϵ_1^1, an element numerically smaller than ϵ_1^1 is obtained in the first column and rth row. Then, on subtracting the rth row from (or adding it to) the first row the matrix is converted into one with a smaller non-zero element in place of ϵ_1^1. This has been done by a succession of operations of type (a). Similarly, if there were an element in the first row which did not contain ϵ_1^1 as a factor, transformations of type (b), strictly analogous to those of type (a) just described, could be set up which would reduce the numerical value of ϵ_1^1.

Second, if ϵ_1^1 is a factor of all the elements of the first row and first column, but is not a factor of the element in the rth row and sth column, ϵ_r^s, we proceed as follows. Upon subtracting the first column from (or adding it to) the sth $\epsilon_1^s/\epsilon_1^1$ times (transformations of type (b)), the first element in the sth column becomes zero, while the rth is still not divisible by ϵ_1^1, since it has been changed by a multiple of ϵ_1^1. If we now add the sth column to the first (an operation of type (b)), the element in the first row and column remains ϵ_1^1, while the rth element in the first column is now not divisible by ϵ_1^1. Hence we may replace ϵ_1^1 by a numerically smaller element by the method of the preceding paragraph.

If the element which replaces ϵ_1^1 is not a factor of all the elements of the matrix, it may be still further reduced by a repetition of the process described in the two paragraphs above. If this process be continued, we must arrive after a finite number of steps—the number being less than the absolute value of ϵ_1^1—at a matrix whose first element d_1 is a factor of all the others. When this point is reached, we may reduce all the elements in the first column except the first to zeros by operations of type (*a*), for we have merely to add the first row to (or subtract it from) any other row the number of times the first element of this row contains d_1. The elements of the first row, with the exception of the first, may be reduced to zeros by similar operations of type (*b*). It is evident that all the elements of the matrix thus obtained contain the first element as a factor.

Thus we arrive at a matrix E_3 in which the first element d_1 of the first column is $\pm$ the H. C. F. of all the elements of E_3 and in which all the other elements of the first row and of the first column are zero. By § 6, d_1 is $\pm$ the H. C. F. of all the elements of E.

9. Let $\bar{E}_3$ be the matrix obtained from E_3 by deleting its first row and first column. By § 8, $\bar{E}_3$ may be reduced to a matrix with a leading element which is $\pm$ the H. C. F. of all its elements, and having all the other elements of the first row and column zero.

As the transformations of types (*a*) and (*b*) which effect this reduction on $\bar{E}_3$ determine transformations of E_3 of the same type, which leave its first row and first column unchanged, we may reduce the matrix E_3 to a matrix E_4 in which the first element of the main diagonal, d_1, is $\pm$ the H. C. F. of all the elements of the matrix, the second element of the main diagonal, d_2, is $\pm$ the H. C. F. of all the elements except d_1, and all the remaining elements of the first two rows and first two columns are zero.

By a continuation of this process we arrive by a finite sequence of operations of types (*a*) and (*b*) at a matrix:

$$(5)\qquad E^* = \left\| \begin{matrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & d_r & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{matrix} \right\|,$$

in which all the elements are zero except a sequence of elements d_i, $(0 < i \leqq r)$ common to the ith row and column and such that d_i is $\pm$ the H. C. F. of all d_j's such that $i \leqq j \leqq r$.

The d_i's may be positive or negative integers. We can make all except the last positive by a sequence of operations of type (a). For if d_i is negative, and we interchange the ith and rth rows, changing the sign of the elements in the rth, a permissible operation by § 4, and repeat the process, we arrive at a form in which d_i is positive and d_r has changed sign. We may thus obtain a form in which d_i $(i < r)$ is positive, and d_r will be positive or negative. Now unless E^* is a square matrix whose rank equals the number of its rows, we can also make d_r positive. This is done by interchanging the rth row (or column) with a row (or column) consisting entirely of zeros and changing the signs of the elements in the rth, and then repeating the process. In case the rank of E^* equals the number of its rows and columns, then the sign of d_r is uniquely determined; in fact it is the same as the sign of E, since E and E^* have the same value. Thus we have given unique determinations for the signs of the d_i in E^*. We shall take this matrix, with at most one negative element, as the normal form E^* in the discussion which follows.

Each operation of type (a) amounts, according to § 3, to multiplying the matrix to which it is applied on the left by a square matrix of type A_0 of α rows, and each operation of type (b) amounts to multiplying the matrix to which it is applied on the right by a square matrix of type B_0 of β rows. Hence

$$(6) \qquad E^* = A \cdot E \cdot B,$$

where A is a product of matrices of type A_0 and B a product of matrices of type B_0. It is to be noted that the determinants of A and B are each $+1$.

We note that by allowing -1 as an admissible value of A or of B, we can obtain a reduction to normal form E^* in which every d_i is positive, even in the exceptional case mentioned above. For, supposing d_r is negative, we can replace A by the determinant obtained from A by changing the signs of the elements in its rth row; and the resulting E^* will have its d_r positive. The result can be obtained equally well by changing the signs of all the elements of the rth column in B.

Let us introduce the notation $D_1 = d_1$, $D_2 = d_1 \cdot d_2, \cdots$ $D_r = d_1 \cdot d_2 \cdots d_r$, and observe that except perhaps for sign when $\gamma = r$, $D_\gamma (0 < \gamma \leqq r)$ is the H. C. F. of all the γ-rowed determinants which can be formed by striking out $\alpha - \gamma$ rows and $\beta - \gamma$ columns from E^*. That is, referring to § 6, they are the successive determinant factors of E^*. Since E^* was derived from E by operations of types (a) and (b), they are also the determinant factors of E.

Since the D_i's are invariant under transformations of types (a) and (b), the d_i's, which are the quotients of successive D_i's, ($d_{i+1} = D_{i+1}/D_i$), are also invariant under these transformations. They are called *invariant factors* or elementary divisors.†

The number r is also invariant under transformations of types (a) and (b) and is called the *rank* of the matrix E.

The Matrices of Transformation

10. In the special case where E is a square matrix of α rows whose determinant is $+1$, equation (6) implies that

† We shall use the term invariant factor, following Bôcher, Introduction to Higher Algebra, pp. 269–70, since the term elementary divisor is sometimes used in another sense.

the determinant of E^* is $+1$. Hence $r = \alpha$, and the numbers d_i must be $+1$.

We therefore have:

$$(7) \qquad A \cdot E \cdot B = I,$$

in which A is a product of matrices of type A_0, B a product of matrices of type B_0 and I is the identity matrix. We may write (7) in the form

$$(8) \qquad E = A^{-1} \cdot I \cdot B^{-1} = A^{-1} \cdot B^{-1}.$$

It is evident from § 3 that when $\alpha = \beta$ every matrix of type B_0 can be regarded also as one of type A_0, and the same is true of matrices inverse to those of types A_0 or B_0. As the above equation shows that E is equal to a product of such matrices, we have the theorem: *Any square matrix of determinant unity is expressible as a product of matrices which may be considered to be of type A_0, or to be of type B_0.*

Hence to multiply a matrix E of α rows and β columns on the left by a square matrix of α rows and determinant unity is equivalent to operating on E by a sequence of operations of type (a); and to multiply E on the right by a square matrix of β columns and determinant unity is equivalent to operating on E by a sequence of operations of type (b).

Also, since we may write (8) in either of the forms:

$$(9) \qquad B \cdot A \cdot E = I \quad \text{or} \quad E \cdot B \cdot A = I,$$

it follows that *if E is a square matrix of determinant unity, it may be reduced to the form I by operations on rows only, or by operations on columns only.*

11. In the case of a general matrix E, we have from (6)

$$(10) \qquad A \cdot E = E^* \cdot B^{-1}.$$

Since the determinant of B^{-1} is 1, the H. C. F. of the elements of its first row is 1. Hence the H.C.F. of the elements of the first row of the matrix $E^* \cdot B^{-1}$ is d_1. As a similar statement applies to the remaining rows, we have the theorem:

The matrix A has the property that the H.C.F. of the elements of the rth row of the matrix $A \cdot E$ is d_r, the rth invariant factor of E (except perhaps for sign in the last row).

This suggests a method of building up A by means of the theorems:

(1) That for any set of integers $\epsilon_j^1\,(0 < j \leqq \alpha)$, a set of integers $a_1^j\,(0 < j \leqq \alpha)$ can be found which are relatively prime and such that

$$\sum_{j=1}^{j=\alpha} a_1^j \cdot \epsilon_j^1 = t_1$$

where t_1 is the H.C.F. of the α ϵ_j^1's; and

(2) That there exists a matrix A of determinant unity with the numbers a_1^j as the elements of its first row.

The derivation of equation (6) by this method is longer than that given in §§ 8 and 9 and is therefore omitted.

Diophantine Equations of the First Degree

12. Consider the problem of finding the integral solutions of the following set of equations:

$$(11) \qquad \begin{aligned} \epsilon_1^1 x_1 + \epsilon_1^2 x_2 + \cdots + \epsilon_1^\beta x_\beta &= p_1, \\ \epsilon_2^1 x_1 + \epsilon_2^2 x_2 + \cdots + \epsilon_2^\beta x_\beta &= p_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \\ \epsilon_\alpha^1 x_1 + \epsilon_\alpha^2 x_2 + \cdots + \epsilon_\alpha^\beta x_\beta &= p_\alpha. \end{aligned}$$

If X denotes the matrix of one column, and β rows

$$\begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_\beta \end{vmatrix},$$

and P a similar matrix with $p_1, p_2, \cdots, p_\alpha$ as the elements of its one column and α rows, equations (11) may be written

$$(12) \qquad E \cdot X = P.$$

But from (6) we have

(13) $$E = A^{-1} \cdot E^* \cdot B^{-1},$$

and consequently

(14) $$A^{-1} \cdot E^* \cdot B^{-1} \cdot X = P,$$

or

(15) $$E^* \cdot B^{-1} \cdot X = A \cdot P.$$

Let us set $Q = A \cdot P$, a matrix of one column and α rows, and denote its elements by $q_1, q_2, \cdots, q_\alpha$. Also let $y_1, y_2, \cdots, y_\beta$ be the elements of the matrix $Y = B^{-1} \cdot X$, which is of one column and β rows. Then (15) becomes:

(16) $$E^* \cdot Y = Q,$$

which is equivalent to the set of α equations:

(17) $$\begin{aligned} d_i y_i &= q_i \qquad && (0 < i \leqq r), \\ 0 &= q_j && (r < j \leqq \alpha), \end{aligned}$$

where r is the rank of E. If equations (17) are to be consistent, the q_j's must all be zero, and in this case the solution is:

(18) $$\begin{aligned} y_i &= \frac{q_i}{d_i} \qquad && (0 < i \leqq r), \\ y_j &\text{ is arbitrary} && (r < j \leqq \beta). \end{aligned}$$

Since the equations $Y = B^{-1} \cdot X$ have integral solutions $x_1, x_2, \cdots, x_\beta$ if and only if the y's are integers, a necessary and sufficient condition that (11) be solvable in integers is that the y's be integers.

To express a condition that equations (17) be consistent and *solvable in integers*, in terms of the coefficients of (11), we proceed as follows. Form the "augmented matrix" of the system, a matrix $\bar{E}$ of α rows and $\beta + 1$ columns whose ith row has as its elements:

$$\varepsilon_i^1, \varepsilon_i^2, \cdots, \varepsilon_i^\beta, -p_i.$$

The matrix $\bar{S}$ formed by multiplying $\bar{E}$ on the left by A will have as the elements of its ith row $(0 < i \leqq \alpha)$:

$$s_i^1, s_i^2, \cdots, s_i^\beta, -q_i,$$

where the s_i^j's are the elements of the matrix:

$$S = \|s_i^j\| = A \cdot E.$$

Since multiplying the matrix S by B reduces it to the normal form, it may be reduced to the form E^* by operations on columns only; which shows that $\bar{S}$ may be reduced by operations on columns only to the form:

$$(19)\quad \bar{E}^* = \left\| \begin{array}{ccccccccc} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -q_1 \\ 0 & d_2 & \cdots & 0 & 0 & \cdots & 0 & -q_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & d_r & 0 & \cdots & 0 & -q_r \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -q_\alpha \end{array} \right\|.$$

In order that (11) be solvable at all, we found that q_i must be zero for values of i greater than r. This shows that the rank of $\bar{E}^*$ is r. If in addition we require the solutions to be integers, q_i must be divisible by d_i for $i \leqq r$. Hence $\bar{E}^*$ may be reduced to normal form by adding the ith column to the last q_i/d_i times. Hence its invariant factors must be the same as the d_i's, i. e., those of E. Conversely, if this condition is satisfied, each q_i will be divisible by the corresponding d_i, and the solutions of (11) will be integers.

Since $\bar{E}^*$ was obtained from $\bar{E}$ by elementary transformations, it has the same rank and invariant factors as $\bar{E}$. Hence we have proved the two theorems:

A necessary and sufficient condition that the equations (11) *have a set of integral solutions is that the augmented matrix* $\bar{E}$ *have the same rank and invariant factors as the matrix of the coefficients* E.

13. Since the solutions of (17) are given by (18), and since $X = B \cdot Y$, the solutions of (11) are:

$$(20) \qquad x_i = \sum_{j=1}^{j=\beta} b_i^j y_j = \sum_{j=1}^{j=r} b_i^j \frac{q_j}{d_j} + \sum_{j=r+1}^{j=\beta} b_i^j y_j \quad (0 < i \leqq \beta).$$

in which $y_{r+1}, y_{r+2}, \cdots, y_\beta$ are arbitrary integers.

If the equations were homogeneous, the p_i's would all be zero, and hence the q_i's would also be zero. Hence the solutions would be of the form:

$$(21) \qquad x_i = \sum_{j=r+1}^{j=\beta} b_i^j y_j \qquad (0 < i \leqq \beta),$$

in which $y_{r+1}, y_{r+2}, \cdots, y_\beta$ are arbitrary integers.

Consequently, for such equations we have the theorem:

A set of linear homogeneous equations whose coefficients are integers has a set of $\beta - r$ linearly independent solutions each of which is a set of relatively prime integers, if β is the number of unknowns and r the rank of the matrix of the coefficients. All other solutions in integers are linearly dependent on these $\beta - r$ linearly independent solutions, the coefficients of the linear relations being integers.

This result was to be expected, since if a set of linear homogeneous equations are solvable in rational numbers, they are solvable in integers.

By comparing (20) and (21) we obtain the further result:

If one set of integers satisfying equations (11) *be given, the other solutions are obtained by adding to it the solutions of the homogeneous equations which result when the right members of* (11) *are replaced by zeros.*

The theorems of this paragraph were first given in complete form by H. J. S. Smith,* although he was anticipated to some extent by Heger.†

Skew-Symmetric Matrices

14. A skew-symmetric matrix is one in which

$$(22) \qquad \epsilon_i^j = -\epsilon_j^i.$$

* Smith, H. J. S., On Systems of Linear Indeterminate Equations and Congruences, Philos. Transactions, Vol. 151 (1861), pp. 293 f. Collected Works, XII, pp. 367 ff.

† Heger, Ignaz, Mem. Vienna Academy, Vol. XIV (1858), second part, p. 111.

Let us define as the *conjugate* of a square matrix the matrix obtained from it by interchanging rows and columns. It is easily verified that if a skew-symmetric matrix be pre-multiplied by any square matrix, and post-multiplied by the conjugate of this matrix, it will remain skew-symmetric.

If A is the matrix defined in § 9 such that

$$(6) \qquad A \cdot E \cdot B = E^*,$$

the matrix $A \cdot E$, by § 11, has d_1 as the H. C. F. of the elements of the first row. Since multiplication on the right corresponds to operations on columns only and leaves the H. C. F. of the elements of the first row unchanged, d_1 is also the H. C. F. of the elements of the first row of $A \cdot E \cdot A'$ where A' denotes the conjugate of A. Since $A \cdot E \cdot A'$ is skew-symmetric, d_1 will also be the H. C. F. of the elements of its first column.

We reduce $A \cdot E \cdot A'$ further as follows: If the second element of the first row does not divide all the remaining elements in that row, let ϵ_1^j be one which it does not divide. Subtract the second column from (or add it to) the jth a number of times equal to the greatest integer in the quotient $\epsilon_1^j/\epsilon_1^2$, thus replacing ϵ_1^j by an element numerically less than ϵ_1^2. Upon subtracting the jth column from (or adding it to) the second, we obtain an element in the first place of the second column smaller than the one there before. All these operations leave the first column unchanged, and since the matrix was skew-symmetric, a similar set of operations on the rows reduces the matrix to a skew-symmetric matrix with the first element in the second column numerically smaller than before. By repeating these operations a sufficient number of times—at most $|\epsilon_1^2|$ times—this first element will be the highest common factor of the elements in the first row, and consequently of the elements of the matrix. But under certain circumstances the resulting first element will be negative. However, by the method of § 9 we can still make it positive (and the first element of the second row then negative).

When this condition is reached, we combine the second column with the other columns such a number of times that all the elements in the first row after the second will be zero, and perform similar operations on the rows. Then we combine the first column with the other columns such a number of times that all the elements in the second row after the first will be zero, and perform similar operations on the rows. This will reduce our matrix to the form:

$$(23) \qquad E_1 = \begin{Vmatrix} 0 & d_1 & 0 & 0 & \cdots & 0 \\ -d_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \epsilon_3^4 & \cdots & \epsilon_3^\alpha \\ 0 & 0 & \epsilon_4^3 & 0 & \cdots & \epsilon_4^\alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \epsilon_\alpha^3 & \epsilon_\alpha^4 & \cdots & 0 \end{Vmatrix}.$$

The matrix obtained from E_1 by deleting its first two rows and columns is skew-symmetric, and by applying the above process to it, in a way wholly analogous to the way by which we extended our initial process of reduction in § 9, we may, by a finite number of operations, reduce our matrix to one, $\overline{E} = \|\epsilon_i^j\|$, in which

$$\epsilon_{2i-1}^{2i} = d_{2i-1}; \qquad \epsilon_{2i}^{2i-1} = -d_{2i-1} \quad (0 < i \leqq p)$$

and the remaining elements are zero. That is, our matrix consists of a series of skew blocks of two non-zero elements each along the main diagonal, surrounded by zero elements. Here all the d_{2i-1} will be positive, except possibly the last, for which the method used above to make it positive will not work if the rank of E equals the number of its rows and columns. It is an interesting question whether the sign of d_{2p-1} is uniquely determined in this case. It is easily made positive if we do not restrict ourselves to symmetrical operations on rows and columns.

Since in the above process we have always performed identical operations on rows and columns, we may write:

(24) $$E = U \cdot \bar{E} \cdot U'$$

where U and U' are conjugate matrices whose determinants are $+1$.

Since interchanging the first and second, third and fourth, $\cdots$ $(2p-1)$th and $2p$th rows, and changing the signs of the even rows would reduce this matrix to the usual normal form E^*, the d_i's appearing in $\bar{E}$ must be identical with those of E^*, i. e., the invariant factors of E (except that the signs of the last two d_i's may come out wrong). Hence, we have the result:

The invariant factors of a skew-symmetric matrix are equal in pairs, and the rank of such a matrix is an even number. A skew-symmetric matrix may be reduced to the "skew" normal form, $\bar{E}$, by multiplying on the left by a unimodular matrix U and on the right by its conjugate, U'.

Symmetric Matrices

15. A *symmetric* matrix is one in which:

(25) $$\varepsilon_i^j = \varepsilon_j^i.$$

Since a symmetric matrix retains its symmetry when we perform any operations on its rows, provided we perform the same operations on its columns, the question naturally arises whether a process similar to that of the preceding paragraph exists which will enable us to reduce such matrices to their normal form by means of a matrix and its conjugate. This question must be answered in the negative*, as is proved by the following example. The matrix

(26) $$E = \begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix},$$

* On p. 189 of Scott and Mathews' Determinants the erroneous statement is made that symmetric matrices with integral elements can always be reduced to normal form by identical operations on rows and columns.

can not be reduced to its normal form,

(27)
$$E^* = \begin{Vmatrix} 1 & 0 \\ 0 & 3 \end{Vmatrix},$$

by a matrix

(28)
$$U = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix},$$

and its conjugate U', where a, b, c and d are integers, since one of the conditions the elements of U would have to satisfy is:

(29)
$$2a^2 + 2ab + 2b^2 = 1.$$

The reduction and classification of symmetric matrices by identical operations on rows and columns is thus a problem of a different order from those which have been considered in this paper. Equivalence under such operations involves much more than the equivalence of invariant factors. This classification is a problem of the arithmetic theory of quadratic forms.*

Matrices with Elements Reduced, Modulo 2

16. In many applications of matrices to Analysis Situs, it is found convenient to reduce the elements of the matrices modulo 2. On reducing modulo 2, the equation (6) becomes

(30)
$$\bar{A} \cdot \bar{E} \cdot B = \bar{E}^*,$$

in which E can represent an arbitrary matrix of α rows and β columns whose elements are 0 and 1, $\bar{A}$ and $\bar{B}$ represent square matrices of determinant unity (mod. 2) of α and β rows respectively, and $\bar{E}^*$ is a matrix all of whose elements are 0 except a sequence of elements along the main diagonal which are 1. This follows from the fact that if one of the invariant factors of E^* is even, so are all the following

* Cf. Encyclopédie des Sciences Mathématiques, Tome I, vol. 3, p. 101.

invariant factors since they contain this one as a factor. The number of 1's in $\bar{E}^*$ is the rank of $\bar{E}^*$. It is less than or equal to the rank of E^*, and differs from it by the number of even invariant factors of E.

Symmetric Matrices, Modulo 2

17. The theory of symmetric matrices, mod. 2, is not subject to the difficulties referred to in § 15. The reduction of such a matrix to normal form may be effected as follows: First interchange rows (performing the same interchange of columns) until the main diagonal consists of a series of 1's followed by a series of 0's. This can be effected by elementary transformations according to § 4 because the negative of any element is the same as the element itself, modulo 2. Add the first row to every row whose first element is a 1, and the first column to the corresponding columns. Repeat this for the second row and column performing a new interchange of rows and columns, if necessary, and continue until there are no elements different from zero in the main diagonal after those used.

The part of the matrix still to be normalized is now in the skew-symmetric form, since $+1 = -1$ (mod. 2) and may be normalized by the process of § 14. Thus by identical operations on rows and columns we have reduced our matrix to the form $\bar{E}^* = \|\epsilon_i^j\|$ in which:

$$\epsilon_i^i = 1 \, (0 < i \leqq p); \quad e_{p+2i-1}^{p+2i} = 1; \quad e_{p+2i}^{p+2i-1} = 1 \, (0 < i \leqq q),$$

and all the remaining elements of the matrix are zero. That is, the non-zero elements consist of a series of 1's in the main diagonal, followed by a series of skew blocks, each containing two 1's. If $p = 0$, this matrix can not be reduced further; but if $p \neq 0$, it may be reduced to a form containing one or two 1's in the main diagonal (according as p is odd or even) and a series of skew blocks, or to a form containing a series of 1's in the main diagonal and no skew blocks. This further reduction depends on the fact that a group of

three 1's in the main diagonal of a matrix in the above form may be replaced by a single 1 in the main diagonal and a skew block of two.

The steps of the process in the case of a three-rowed square matrix are, first adding the first row and column to the second row and column respectively, then adding the third row and column to the first row and column respectively, and finally adding the second row and column to the third. The matrix becomes successively:

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

These steps may be made in reverse order to effect the inverse transformation, and are obviously typical of the steps which can be applied to any matrix $\bar{E}^*$ for which $p > 0$.

INDEX

Alexander, J. W., v, vii, 42, 61, 71, 138, 154, 155, 158.
Analysis Situs, 3, 37, 51, 77.
Anchor ring, 48.
Betti numbers, 117; relation to connectivity numbers, 120; invariance of, 128–130; duality of, 130–131.
Birkhoff, G. D., 43.
Boundary: of a 1-cell, 1; of an open curve, 13; of a one-dimensional complex, 20; of an oriented 1-cell, 31; of a 2-cell, 36; of a 2-dimensional complex, 46; of an n-cell, 76; of a k-dimensional complex, 79, 112; of an oriented complex, 108.
Brahana, H. R., 75, 157.
Briggs, G. B., 158.
Brouwer, L. E. J., 88, 135.
Brown, A. B., v.
Cell: 0-cell, 1; 1-cell, 1; 2-cell, 37; n-cell, 76, 86.
Characteristic, 9, 42, 79, 82.
Circuit: 0-circuit, 13–15; 1-circuit, 16–17; oriented 1-circuit, 25, 28; bounding and non-bounding 1-circuits, 57; 2-circuit, 47; invariance of 2-circuit, 67; n-circuit, 77; bounding and non-bounding k-circuits, 98; invariance of n-circuit, 103; circuit is a singular generalized manifold, 96.
Coefficients of torsion, 119; invariance of, 128–130; duality of, 130–131.
Complete set, 15; of 1-circuits, 18; of non-bounding 1-circuits, 53; of sets of oriented k-circuits, 114–115; of non-bounding sets of oriented k-circuits, 114–116; of sets of oriented k-circuits each of which bounds or has a multiple that bounds, 114–115.
Complex: zero-dimensional, 2, (singular) 5-6; one-dimensional, 2, (generalized) 2, (singular) 5–6, 30; two-dimensional, 36, (generalized) 37, (singular) 55—56; n-dimensional, 76, (generalized) 77, (singular) 77; regular, 88.
Congruence, 58, 84, 121.
Connectivity numbers: R_0, 7; R_1, 50–52; invariance of R_1, 65; R_k, 81; duality of, 95; invariance of, 103.
Correspondence A, 61, 98.
Cover (one complex by another), 6, 56, 106–107.
Covering manifold, 137, 152–154.

Curve: open, 3; closed, 4.
Cyclomatic number, 9.
Cylindrical surface, 48.
Deformation, 132; homotopic, 133; isotopic, 133.
Dehn, M., vii, 9, 133, 152; 154, 156, 157, 158.
Determinant factor, 173.
Diophantine equations, 180–183.
Dual complexes, 92.
Duality: of connectivities, 95; of Betti numbers, 130–131; of torsion coefficients, 130–131.
Dyck, W., 9.
Equivalence, 148, 153.
Four-color problem, 43.
Franklin, P., v, vii, 43, 171.
Fundamental congruences and homologies, 85, 123.
Generalized manifold, 95–96.
Generating relations, 143, 145–147.
Generators of a group, 142–143, 146.
Genus, 50–51.
Gieseking, H., 152.
Group: fundamental, 139–140; of a linear graph, 141–143; of a two-dimensional complex, 143–145; commutative, 145–148; homology, 149.
Hahn, L., 87.
Heegaard, P., vii, 9, 133, 138, 156, 157.
Heegaard diagram, 155–157.
Heger, I., 183.
Heine-Borel theorem, 5.
Homeomorphism, 3, 37, 77.
Homology, 59, 84, 121.
Homotopy, 133, 138–139.
Incidence, 4, 37, 77.
Indicatrix, 136.
Intersection number: of a k-cell and an $(n-k)$-cell, 160, 162; of a k-dimensional complex and an $(n-k)$-dimensional complex, 162–163; of the fundamental complexes, 166; of sets of k-circuits with sets of $(n-k)$-circuits, 167; modulo 2, 169.
Invariant factor, 178.
Isotopy, 133.
Jordan, C., 157.
Jordan theorem, 41, 58, 88.
Kirchhoff, G., 9, 26.
Knot problem, 158.
König, D., 71, 111.
Kronecker, L., 159.
Lefschetz, S., v.

Lennes, N. J., 86.
Linear graph, 2, 26, 141–143.
Linkage, 167.
Listing J. B., 9.
Manifold: 0-dimensional, 13; two-dmensional, 47, (orientable) 50, 73; normal form for two-dimensional, 74; n-dimensional, 92, (generalized) 95–96; covering, 137, 152–154; three-dimensional, 154–157.
Map, 42–43.
Matrices of incidence, 11, 12, 37, 38; normal form for, 22, 82, 84.
Matrices of orientation, 25–27, 68, 105; normal form for, 32–33, 113, 117.
Matrix, 170; product of two, 171; inverse of, 171; elementary transformation of, 171–173; reduction to normal form, 174–178; skew-symmetric, 183–186; conjugate of, 184; symmetric, 186; with elements reduced mod. 2, 187, (symmetric) 188–189.
Möbius, A. F., 49, 71.
Möbius strip, 49, 135.
Morse, M., v.
Neighborhood, 6, 47, 91.
Nielsen, J., v, 135, 152, 157.
One-sided, see Orientable.
Orientable 2-circuit, 71; invariance of, 73.
Orientable n-circuit, 109; invariance of, 110.
Orientation: of a 0-cell, 23; of a 0-circuit, 24; of a 1-cell, 24; of a 1-dimensional complex, 24; of a 1-circuit, 25; of an n-cell, 104–105; of an n-dimensional complex, 105; of a singular k-dimensional complex, 106; of a cell by association with a sense-class, 134; of a regular cell by the order of its vertices, 135.
Oriented n-circuit, 109; singular, 110; bounding and non-bounding, 126.
Pfeiffer, G. A., v.
Poincaré, H., vi, vii, 2, 25, 117, 120, 122, 148, 154, 157, 159, 162.
Poincaré numbers, 151–152; invariance of, 151; are the one-dimensional coefficients of torsion, 151.
Poincaré space, 154, 158.
Projective plane, 50–51, 153.
Regular complex, 44, 88.
Regular subdivision, 43, 89.
Riemann space, 158.
Riemann surface 137–138.
Schoenflies theorem, 41, 58, 88, 155.
Sense-class, 134–135.
Similarly oriented, 135.
Simplex, 1, 36, 76.
Simply-connected map, 42.
Singular point, 5, 56.
Smith, H. J. S., 183.

Sphere, 48, 79.
Star of cells, 88; simply-connected, 88.
Steinitz, E., 71, 137.
Sum: mod. 2, 10; mod. 2, of two complexes, 10, 50, 79; of two oriented complexes, 105, 107.
Tait, P. G., 158.
Tetrahedron, 3.
Tietze, H., 71, 120, 135, 151.
Torsion, see Coefficients of torsion.
Torus, 48.
Tree, 18, 47.
Triangle, 43.
Triangle star, 44.
Universal covering surface, 153.
Veblen, O., 43, 50, 86.
Vertex, 1, 36, 76.
Weyl, H., 153.

COLLOQUIUM PUBLICATIONS

Volume I. H. S. WHITE, Linear Systems of Curves on Algebraic Surfaces; Forms of Non-Euclidean Space; F. S. WOODS, Forms of Non-Euclidean Space; E. B. VAN VLECK, Topics in the Theory of Divergent Series and of Continued Fractions. 1905. 12 + 187 pp. \$2.75.

Volume II. Out of Print.

Volume III. G. A. BLISS, Fundamental Existence Theorems; EDWARD KASNER, Differential-Geometric Aspects of Dynamics. 1913. 6 + 226 pp. \$2.50.

Volume IV. L. E. DICKSON, On Invariants and the Theory of Numbers; W. F. OSGOOD, Topics in the Theory of Functions of Several Complex Variables. 1914. 4 + 230 pp. \$2.50.

Volume V, PART I. G. C. EVANS, Functionals and their Applications, Selected Topics including Integral Equations. 1918. 12+136 pp. \$2.00.

Volume V, PART II. OSWALD VEBLEN, Analysis Situs. Second edition, 1931. 10 + 194 pp. \$2.00.

Volume VI. G. C. EVANS, The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems. 1927. 8 + 150 pp. \$2.00.

Volume VII. E. T. BELL, Algebraic Arithmetic. 1927. 4 + 180 pp. \$2.50.

Volume VIII. L. P. EISENHART, Non-Riemannian Geometry. 1927. 8 + 184 pp. \$2.50.

Volume IX. G. D. BIRKHOFF, Dynamical Systems. 1927. 8 + 295 pp. \$3.00.

Volume X. A. B. COBLE, Algebraic Geometry and Theta Functions. 1929. 8 + 282 pp. \$3.00.

Volume XI. DUNHAM JACKSON, The Theory of Approximation. 1930. 8 + 178 pp. \$2.50.

Volume XII. S. LEFSCHETZ, Topology. 1930. 10 + 410 pp. \$4.50.

Orders may be addressed to

AMERICAN MATHEMATICAL SOCIETY
501 West 116th Street
NEW YORK CITY

or to any of the following official agents of the Society:

BOWES & BOWES, 1-2 Trinity Street, Cambridge, England;

HIRSCHWALDSCHE BUCHHANDLUNG, Unter den Linden 68, Berlin, Germany;

LIBRAIRIE SCIENTIFIQUE ALBERT BLANCHARD, 3 Place de la Sorbonne, Paris 5, France;

LIBRERIA EDITRICE NICOLA ZANICHELLI, Bologna, Italy.